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A STUDY ON A THEOREM OF HAMEL

A THESIS

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by

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CORRIGENDA

Type of Errors: (a) Typographical errors, (b) Copying errors, (c) Incorrect re-arranged forms, which are not originally used in calculations.

Page	Line	Incorrect	Correct	Type of Error
5	8	curl $\underset{\sim}{V}$ = grad V x $\underset{\sim}{s}$ + V curl $\underset{\sim}{s}$.	curl $\underset{\sim}{V}$ = grad V x $\underset{\sim}{s}$ + V curl $\underset{\sim}{s}$ = 0 .	(b)
7	17	From (1.15), (1.16), and (1.17),	From (2.15), (2.16), and (2.17),	(b)
8	5	From (1.4) and (1.18)	From (2.4) and (2.18)	(b)
9	16	Applying (2.28) and (2.30) to V,···	Applying (2.29) and (2.31) to V,···	(b)
11	13	of the $\underset{\sim}{b}$-lines on···	of the b-lines on···	(b)
23	7	$\frac{\delta^2 \varkappa_s}{\delta s \delta b} = -\frac{3\tau_2}{\tau_{\bar{s}}^2}\left(\tau_{\bar{s}}^2 + \cdots\right.$	$\frac{\delta^2 \varkappa_s}{\delta s \delta b} = -\frac{3\tau_s}{\tau_{\bar{s}}^2}\left(\tau_{\bar{s}}^2 + \cdots\right.$	(a)
28	9	···for $\frac{\delta\theta_{bs}}{\delta b}$, $\varkappa$ + div $\underset{\sim}{n}$ and div $\underset{\sim}{b}$,···	···for $\frac{\delta\theta_{bs}}{\delta b}$, $\varkappa_s$ + div $\underset{\sim}{n}$ and div $\underset{\sim}{b}$,···	(a)
33	4	$\cdots + \frac{\theta_{bs}}{\tau_s \varkappa_s}\left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right)$div $\underset{\sim}{b}$	$\cdots + \frac{\theta_{bs}}{\tau_s \varkappa_s}\left(3\tau_{\bar{s}}^2 - \frac{\varkappa_s^2}{4}\right)$div $\underset{\sim}{b}$	(c)

Page	Line	Incorrect	Correct	Type of Error
34	14	$+\frac{8\tau_s\theta_{bs}}{\varkappa_s}\Bigl(\tau_s^{\;2} - \frac{\varkappa_s^{\;2}}{4}\Bigr)\,, \qquad (4.11)$	$+\frac{8\tau_s\theta_{bs}}{\varkappa_s}\Bigl(\tau_{\bar{s}}^{\;2} - \frac{\varkappa_s^{\;2}}{4}\Bigr)\,, \qquad (4.11)$	(a)
35	7	$-\frac{4\tau_s\theta_{bs}}{\tau_{\bar{s}}^{\;2}\varkappa_s^{\;2}}\Bigl[4\tau_{\bar{s}}^{\;4} + 2\tau_s^{\;2}\tau_{\bar{s}}^{\;2} + \cdots$	$-\frac{4\tau_s\theta_{bs}}{\tau_{\bar{s}}^{\;2}\varkappa_s^{\;2}}\Bigl[\tau_{\bar{s}}^{\;4} + 2\tau_s^{\;2}\tau_{\bar{s}}^{\;2} + \cdots$	(b)
36	4	$-\frac{1}{\tau_{\bar{s}}^{\;2}\varkappa_s^{\;2}}\Bigl(\tau_{\bar{s}}^{\;6} + (8\tau_s^{\;2} + \cdots$	$-\frac{1}{\tau_{\bar{s}}^{\;2}\varkappa_s^{\;2}}\Bigl[\tau_{\bar{s}}^{\;6} + \Bigl(8\tau_s^{\;2} + \cdots$	(a)
36	8	$-\frac{\varkappa\,\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}}\frac{\delta\tau_s}{\delta s} + \cdots$	$-\frac{\varkappa_s\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}}\frac{\delta\tau_s}{\delta s} + \cdots$	(a)
37	10	$\cdots + \frac{4\theta_{bs}}{\varkappa_s^{\;2}}\Bigl(\tau_{\bar{s}}^{\;2} + \frac{\varkappa_s^{\;2}}{4}\Bigr)\Bigr]\frac{\delta\varkappa_s}{\delta s}$	$+\frac{2\theta_{bs}^{\;2}}{\varkappa_s^{\;2}}\Bigl(\tau_{\bar{s}}^{\;2} + \frac{\varkappa_s^{\;2}}{4}\Bigr)\Bigr]\frac{\delta\varkappa_s}{\delta s}$	(c)
46	14	$-\Bigl[\frac{2\tau_{\bar{s}}^{\;2}\varkappa_s}{\theta_{bs}^{\;3}} - \frac{\tau_s}{\theta_{bs}^{\;2}}\Bigl(2\tau_{\bar{s}}^{\;2} - \cdots$	$+\Bigl[\frac{2\tau_{\bar{s}}^{\;2}\varkappa_s}{\theta_{bs}^{\;3}}\,\mathrm{div}\,\underset{\sim}{b} - \frac{\tau_s}{\theta_{bs}^{\;2}}\Bigl(2\tau_{\bar{s}}^{\;2} - \cdots$	(c)
46	15	$\cdots - \frac{\tau_s}{\theta_{bs}^{\;3}}\Bigl(2\tau_s^2 - \frac{\varkappa_s^{\;2}}{4}\Bigr)\mathrm{div}\,\underset{\sim}{b}$	$\cdots - \frac{2\tau_s}{\theta_{bs}^{\;3}}\Bigl(2\tau_s^2 - \frac{\varkappa_s^{\;2}}{4}\Bigr)\mathrm{div}\,\underset{\sim}{b}$	(c)

Page	Line	Incorrect	Correct	Type of Error
47	5	$\cdots + \left(\frac{\theta_{bs}}{\tau_s^2}\frac{\delta\tau_s}{\delta n} - \frac{1}{\tau_n}\frac{\delta\theta_{bs}}{\delta n}\right)\frac{\delta\theta_{bs}}{\delta n}$	$\cdots + \left(\frac{\theta_{bs}}{\tau_s^2}\frac{\delta\tau_s}{\delta n} - \frac{1}{\tau_s}\frac{\delta\theta_{bs}}{\delta n}\right)\frac{\delta\theta_{bs}}{\delta n}$	(a)
51	7	$-\left(5\ \mathrm{div}\ \underset{\sim}{b} - \frac{\delta\tau_s\theta_{bs}}{\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta b}\cdots$	$-\left(5\ \mathrm{div}\ \underset{\sim}{b} - \frac{8\tau_s\theta_{bs}}{\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta b}$	(a)
51	9	$\cdots - \varkappa_s^2\left(\tau_s^2 - \frac{\varkappa_s}{16}\right)\Big].$	$\cdots - \varkappa_s^2\left(\tau_s^2 - \frac{\varkappa_s^2}{16}\right)\Big].$	(a)
53	9	$-\Big[\frac{2\tau_{\bar{s}}^2\varkappa_s}{\tau_s\theta_{bs}^2} - \frac{1}{\theta_{bs}}\Big(2\tau_{\bar{s}}^2\cdots$	$+\Big[\frac{2\tau_{\bar{s}}^2\varkappa_s}{\tau_s\theta_{bs}^2}\ \mathrm{div}\ \underset{\sim}{b} - \frac{1}{\theta_{bs}}\Big(2\tau_{\bar{s}}^2\cdots$	(c)
53	10	$-\frac{1}{\theta_{bs}^2}\left(2\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\mathrm{div}\ \underset{\sim}{b} + \cdots$	$-\frac{2}{\theta_{bs}^2}\left(2\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\mathrm{div}\ \underset{\sim}{b} + \cdots$	(c)
53	10	$\cdots + \frac{2\theta_{bs}}{\tau_s\varkappa_s}\left(\tau_{\bar{s}}^2 + 2\theta_{bs}^2\frac{\varkappa_s^2}{4}\right)\frac{\delta\theta_{\varkappa}}{\delta b}$	$\cdots + \frac{2\theta_{bs}}{\tau_s\varkappa_s}\left(\tau_s^2 + 2\theta_{bs}^2 - \frac{\varkappa_s^2}{4}\right)\frac{\delta\theta_{bs}}{\delta b}$	(a)
63	14	$\cdots - \left(\frac{\mathrm{div}\ \underset{\sim}{b}}{\theta_{bs}} - \frac{8\tau_s^2\theta_{bs}}{\tau_{\bar{s}}^4\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta b}$	$\cdots - \left(\frac{\mathrm{div}\ \underset{\sim}{b}}{\theta_{bs}^2} - \frac{8\tau_s^3\theta_{bs}}{\tau_{\bar{s}}^4\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta b}$	(c)

Page	Line	Incorrect	Correct	Type of Error
64	1	$\cdots + \frac{\delta\tau_s\theta_{bs}^3}{\tau_{\bar{s}}^4\varkappa_s}\frac{\delta\tau_s}{\delta b}\Big]\frac{\delta\theta_{bs}}{\delta b}$	$\cdots + \frac{8\tau_s\theta_{bs}^3}{\tau_{\bar{s}}^4\varkappa_s}\frac{\delta\tau_s}{\delta b}\Big]\frac{\delta\theta_{bs}}{\delta b}$	(a)
64	6	$\cdots - \left(\frac{\operatorname{div}\underset{\sim}{b}}{\theta_{bs}^2} + \frac{8\tau_s^3\theta_{bs}}{\tau_s^4\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta n}$	$\cdots - \left(\frac{\operatorname{div}\underset{\sim}{b}}{\theta_{bs}^2} + \frac{8\tau_s^3\theta_{bs}}{\tau_{\bar{s}}^4\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta n}$	(a)
66	5	$\frac{\delta^2}{\delta s\delta b}\operatorname{div}\underset{\sim}{b} = -\frac{\delta^2}{\delta s\delta n}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) + \cdots$	$\frac{\delta^2}{\delta s\delta b}\operatorname{div}\underset{\sim}{b} + \frac{\delta^2}{\delta s\delta n}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) + \cdots$	(b)
67	12	$\cdots + \frac{\delta\tau_s\theta_{bs}}{\varkappa_s^2}\left(\tau_{\bar{s}}^2 - \frac{\varkappa_s^2}{4}\right).$	$\cdots + \frac{8\tau_s\theta_{bs}}{\varkappa_s^2}\left(\tau_{\bar{s}}^2 - \frac{\varkappa_s^2}{4}\right)$	(a)
68	7	$\frac{\delta^2}{\delta b\delta s}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) = -\frac{\delta^2\theta_{bs}}{\delta b\,\delta n} + \cdots$	$\frac{\delta^2}{\delta b\delta s}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) + \frac{\delta^2\theta_{bs}}{\delta b\delta n} + \cdots$	(b)

Lemma 6.1 is incorrect, because (6.4) and (6.7) are equivalent conditions, each reducing to (6.5)

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SUMMARY

A theorem on potential flows was given by Hamel in 1937, with much of its proof omitted. Although Hamel's result is highly plausible neither a complete proof nor a counter example has since been produced.

In this dissertation an attempt is made to prove the theorem through the approach of differential geometry, which was, in fact, envisaged by Hamel as an alternative method of proof.

It is found in this study that one can express all the gradients of four parameters, which govern the vector field geometry, as rational polynomials. One is then able to derive three polynomial relations among the four parameters as first suspected by Hamel (1937 [1]). One such relation was also mentioned by Howard (1953 [1]).

If it is possible to deduce from the three derived polynomial relations two equations, each of which involves a certain pair of the four parameters, then Hamel's theorem will follow by Lemma 6.1.

However the three polynomial relations are highly complicated, and it remains to be shown whether the two aforementioned equations can always be deduced from these polynomial relations.

CHAPTER I

INTRODUCTION

More than thirty years ago Hamel (1937 [1]) published an analysis establishing the following theorem:

HAMEL'S THEOREM

Let $\underset{\sim}{V} = V(x^\alpha)\underset{\sim}{s}(x^\alpha)$, $\alpha = 1, 2, 3$, be a steady vector field in three-dimensional space*, such that $\underset{\sim}{V}$ is both lamellar and solenoidal, and the magnitude $V(x^\alpha)$ of $\underset{\sim}{V}$ maintains a constant value along a vector-line, then the vector field of $\underset{\sim}{V}$ must either consist of parallel straight lines or circular helices mounted on concentric circular cylinders.

This result specifies completely the vector fields and steady motions defined by the conditions

$$\operatorname{div} \underset{\sim}{V} = 0 \ , \quad \underset{\sim}{s} \cdot \operatorname{grad} \underset{\sim}{V} = 0 \ , \quad \operatorname{curl} \underset{\sim}{V} = 0 \ . \tag{1.1}$$

Furthermore, Prim (1952 [1]) proved that a steady, solenoidal, complex-lamellar, circulation-preserving motion would have the same stream-line geometry as the vector field (1.1). Hamel's result is also the final word on these motions.

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*The symbols x^α, $\alpha = 1, 2, 3$, denote Cartesian co-ordinates, and $\underset{\sim}{s}(x^\alpha)$ is the unit vector tangent to the vector-line.

The result is remarkable for several reasons. As Howard (1953 [1]) pointed out, the two-dimensional case, the only solution is a vector field consisting of concentric circles or parallel straight lines. It is surprising indeed that the further dimension adds little more complication than a superposition of the results for two dimensions.

While the theory of three families of surfaces intersecting orthogonally is classical, forming as it does the basis for the construction of coordinate lines in three dimensional space, for less is known about the properties of two families of surfaces intersecting orthogonally. Hamel's Theorem relates to such systems. Consider a one parameter family of minimal surfaces whose orthogonal trajectories are geodesics on a second family of surfaces intersecting the minimal surfaces orthogonally. Suppose further that the curvature of the orthogonal trajectories is inversely proportional to the distance between the surfaces on which they are geodesics. Then these trajectories must be circular helices mounted on circular cylinders. The minimal surfaces must be right helicoids, the surfaces represented by spiral stair-cases. This is Hamel's Theorem in geometric form.

Hamel's proof of the theorem is not explicit. It is based upon the Weierstrassian representation of minimal surfaces as complex integrals, and depends upon an analysis showing that a certain non-linear partial differential equation has no solutions other than a trivial one.

Hamel's calculations, requiring at one stage no less than 81 separate proofs of contradictions, were so extensive that they had to be omitted almost in their entirety. Thus it has been impossible for others to follow and check Hamel's analysis.

In the final sections of his paper, Hamel seeks to prove the theorem using the intrinsic equations of the vector field geometry. He fails to achieve the result on this basis. His analysis terminates at his derivation of the conditions given here as (2.53) and (2.54). We shall see that all that is required to prove the result is to establish that one parameter called θ_{bs} must vanish. This was understood by Hamel. His final comment in the paper is the statement that he suspects that there exists a second higher order integrability condition which, taken with the one found [condition (2.53)], would prove the result. He was unable to discover this condition.

The preceding discussion indicates that while one must conjecture that Hamel's result is valid, there nevertheless exists a certain doubt of this. Howard, for example, in his thesis of 1953 repeatedly refers to the result "if true".

In this dissertation an attempt is made to prove the theorem through the methods of classical differential geometry, that is, by continuing the method initiated in the last section of Hamel's paper.

As has already been remarked, Hamel suspected the existence of a higher order condition that would prove the result. Three such polynomial relations are discovered, which must be satisfied by four of the vector-field parameters. These conditions are of very high order. It was nevertheless found possible to derive each by two independent approaches. Due to the complexity of the equations, such independent checks were essential in order that arithmetical errors could be eliminated.

The three polynomials to be satisfied by the four vector-field parameters θ_{bs}, τ_s, $\varkappa_s$, div $\mathfrak{b}$ of the exceptional vector field are presented.* It is shown in the Appendix that these are the only three polynomials obtained from the compatibility equations.

In the derivation of the polynomials much of the algebraic computation is omitted. However no steps are omitted in arguments proving theorems.

It is reiterated that in Hamel's paper, details of proofs were omitted, for example the 81 contradictions referred to above. Thus it is believed that this analysis is more explicit than Hamel's, in that all the results given here can be reproduced by following the steps pointed out herein.

It is also shown that if it is possible to deduce from the three derived polynomial relations, two relations of the form $F(\theta_{bs}, \varkappa_s) = 0$, $G(\tau_s, \varkappa_s) = 0$ which allow one to deduce

$$\text{grad } \theta_{bs} \times \text{grad } \varkappa_s = 0 \ ,$$
$$\text{grad } \tau_s \times \text{grad } \varkappa_s = 0 \ ,$$

then Hamel's theorem follows.

*These parameters are defined in the next chapter.

CHAPTER II

PRELIMINARY ANALYSIS

We consider here the vector fields

$$\underset{\sim}{V} = V(x^{\alpha})\underset{\sim}{s}(x^{\alpha}) , \tag{2.1}$$

defined by the three conditions*

$$\operatorname{div} \underset{\sim}{V} = \underset{\sim}{s}\cdot\operatorname{grad} V + V \operatorname{div} \underset{\sim}{s} = 0 , \tag{2.2}$$

$$\underset{\sim}{s}\cdot\operatorname{grad} V = 0 , \tag{2.3}$$

$$\operatorname{curl} \underset{\sim}{V} = \operatorname{grad} V \times \underset{\sim}{s} + V \operatorname{curl} \underset{\sim}{s} . \tag{2.4}$$

In these relations $\underset{\sim}{s}$ is the unit vector tangent to the vector-line of $\underset{\sim}{V}$. The conditions (2.2) and (2.3) show that

$$\operatorname{div} \underset{\sim}{s} = 0 . \tag{2.5}$$

From (2.3) and (2.5) it follows (1947 [1]) that $\underset{\sim}{s}$ has the representation

$$\underset{\sim}{s} = \operatorname{grad} V \times \operatorname{grad} f . \tag{2.6}$$

*Conditions of smoothness sufficient to accommodate all the operations are assumed at the outset.

Provided the curvature $\varkappa_s$ of the vector-line does not vanish in a neighborhood, the principal normal $\underset{\sim}{n}$ to the vector-line is defined by*

$$\underset{\sim}{s}\cdot \text{grad}\ \underset{\sim}{s} = \frac{\delta \underset{\sim}{s}}{\delta s} = \varkappa_s\, \underset{\sim}{n}\ , \qquad (2.7)$$

and the unit bi-normal $\underset{\sim}{b}$ according to

$$\underset{\sim}{b} = \underset{\sim}{s} \times \underset{\sim}{n}\ . \qquad (2.8)$$

The torsion τ_s is then defined by the Serret-Frenet formulae

$$\frac{\delta \underset{\sim}{b}}{\delta s} = -\ \tau_s\, \underset{\sim}{n}\ , \qquad (2.9)$$

and

$$\frac{\delta \underset{\sim}{n}}{\delta s} = -\ \varkappa_s\, \underset{\sim}{n} + \tau_s\, \underset{\sim}{b}\ . \qquad (2.10)$$

If the curvature $\varkappa_s$ of the vector-line vanishes in a neighborhood, the vector lines lie on the one-parameter family of ruled surfaces $V(x^\alpha)$ = constant. In this case one chooses $\underset{\sim}{n}$ to be normal to the surface, so that

$$\underset{\sim}{n} = \psi\ \text{grad}\ V\ . \qquad (2.11)$$

The unit vector $\underset{\sim}{b}$ is then defined as before by (2.8).

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*We use the notation $\frac{\delta}{\delta s}, \frac{\delta}{\delta n}, \frac{\delta}{\delta b}$ to denote the components $\underset{\sim}{s}\cdot$grad, $\underset{\sim}{n}\cdot$grad, $\underset{\sim}{b}\cdot$grad. These operators will be referred to as $\underset{\sim}{s}$-gradient, $\underset{\sim}{n}$-gradient, and $\underset{\sim}{b}$-gradient, respectively. The usual derivative symbol is avoided, since these components are anholonomic.

From (2.4) it is seen that

$$\Omega_s = \mathbf{s}\cdot\operatorname{curl}\mathbf{s} = 0\ . \tag{2.12}$$

One writes

$$\Omega_n = \mathbf{n}\cdot\operatorname{curl}\mathbf{n}\ , \tag{2.13}$$

and

$$\Omega_b = \mathbf{b}\cdot\operatorname{curl}\mathbf{b}\ . \tag{2.14}$$

In accordance with (2.5), (2.7), (2.9), (2.10), and (2.12) one may write*

$$\operatorname{grad}\mathbf{s} = \begin{array}{lll} & +\varkappa_s\,\mathbf{sn} & \\ & -\theta_{bs}\,\mathbf{nn} & -(\Omega_n+\tau_s)\,\mathbf{nb} \\ & -(\Omega_n+\tau_s)\,\mathbf{bn} & +\theta_{bs}\,\mathbf{bb}\ , \end{array} \tag{2.15}$$

$$\operatorname{grad}\mathbf{n} = \begin{array}{lll} -\varkappa_s\,\mathbf{ss} & & +\tau_s\,\mathbf{sb} \\ +\theta_{bs}\,\mathbf{ns} & & -\operatorname{div}\mathbf{b}\,\mathbf{nb} \\ +(\Omega_n+\tau_s)\,\mathbf{bs} & & +(\varkappa_s+\operatorname{div}\mathbf{n})\,\mathbf{bb}\ , \end{array} \tag{2.16}$$

$$\operatorname{grad}\mathbf{b} = \begin{array}{lll} & -\tau_s\,\mathbf{sn} & \\ +(\Omega_n+\tau_s)\,\mathbf{ns} & +\operatorname{div}\mathbf{b}\,\mathbf{nn} & \\ -\theta_{bs}\,\mathbf{bs} & -(\varkappa_s+\operatorname{div}\mathbf{n})\,\mathbf{bn} & \end{array} \tag{2.17}$$

From (1.15), (1.16), and (1.17),

$$\operatorname{curl}\mathbf{s} = \operatorname{grad}\times\mathbf{s} = \varkappa_s\,\mathbf{b}\ , \tag{2.18}$$

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*See the reference (1969 [1]).

$$\text{curl } \underset{\sim}{n} = \text{grad} \times \underset{\sim}{n} = -(\text{div } \underset{\sim}{b})\underset{\sim}{s} + \Omega_n \underset{\sim}{n} - \theta_{bs} \underset{\sim}{b} , \tag{2.19}$$

$$\text{curl } \underset{\sim}{b} = \text{grad} \times \underset{\sim}{b} = (\varkappa_s + \text{div } \underset{\sim}{n})\underset{\sim}{s} - \theta_{bs} \underset{\sim}{n} + \Omega_b \underset{\sim}{b} , \tag{2.20}$$

where

$$\Omega_b = -(\Omega_n + 2\tau_s) . \tag{2.21}$$

From (1.4) and (1.18)

$$\text{curl } \underset{\sim}{V} = \frac{\delta V}{\delta b} \underset{\sim}{n} + \left(\varkappa_s V - \frac{\delta V}{\delta n}\right)\underset{\sim}{b} = 0 , \tag{2.22}$$

so that

$$\frac{\delta V}{\delta b} = 0 , \tag{2.23}$$

and

$$\frac{\delta V}{\delta n} = \varkappa_s V . \tag{2.24}$$

Also,

$$\frac{\delta V}{\delta s} = 0 . \tag{2.3}$$

From (2.24) it is seen that $\frac{\delta V}{\delta n}$ vanishes if and only if $\varkappa_s$ vanishes. The conditions (2.3), (2.23), and (2.24) show that V, assumed to be non-vanishing, is spatially constant if and only if $\varkappa_s$ vanishes. Thus provided $\varkappa_s$ does not vanish in a neighborhood the family of surfaces $V(x^\alpha)$ = constant contains the vector-lines of $\underset{\sim}{s}$ and $\underset{\sim}{b}$. The unit vector $\underset{\sim}{n}$

is parallel to grad V,

$$\underset{\sim}{n} = \psi \text{ grad } V \ . \qquad (2.25)$$

It follows that

$$\Omega_n = \underset{\sim}{n} \cdot \text{curl } \underset{\sim}{n} = 0 \qquad (2.26)$$

a condition which also holds for the case $\varkappa_s$ vanishing in a neighborhood.

From (2.21) and (2.25),

$$\Omega_b = - 2\tau_s \ . \qquad (2.27)$$

In all that follows the conditions (2.15) to (2.20) to be simplified by the conditions (2.25) and (2.26) will be considered.

The identity

$$\text{curl grad } F = 0 \qquad (2.28)$$

applied to the tensor point function F, yields the commutation formulae

$$\frac{\delta^2 F}{\delta b\, \delta n} - \frac{\delta^2 F}{\delta n\, \delta b} = - \text{div } \underset{\sim}{b} \frac{\delta F}{\delta n} + (\varkappa_s + \text{div } \underset{\sim}{n}) \frac{\delta F}{\delta b} \ , \qquad (2.29)$$

$$\frac{\delta^2 F}{\delta s\, \delta b} - \frac{\delta^2 F}{\delta b\, \delta s} = - \theta_{bs} \frac{\delta F}{\delta b} \ , \qquad (2.30)$$

$$\frac{\delta^2 F}{\delta n\, \delta s} - \frac{\delta^2 F}{\delta s\, \delta n} = + \varkappa_s \frac{\delta F}{\delta s} - \theta_{bs} \frac{\delta F}{\delta n} - 2\tau_s \frac{\delta F}{\delta b} \ . \qquad (2.31)$$

Applying (2.28) and (2.30) to V, (2.3), (2.23), and (2.24) yield the relations

$$\frac{\delta \varkappa_s}{\delta b} = - \varkappa_s \operatorname{div} \underset{\sim}{b} , \tag{2.32}$$

$$\frac{\delta \varkappa_s}{\delta s} = \theta_{bs} \varkappa_s . \tag{2.33}$$

The condition (2.32) expresses the condition div curl $\underset{\sim}{s} = 0$, where curl $\underset{\sim}{s}$ is given by (2.18).

The condition

$$\Omega_s = \underset{\sim}{s} \cdot \operatorname{curl} \underset{\sim}{s} = 0 \tag{2.13}$$

shows that vector-lines of $\underset{\sim}{s}$ are the orthogonal trajectories of a one-parameter family of potential surfaces $\varphi(x^\alpha)$ = constant.

The condition

$$\operatorname{div} \underset{\sim}{s} = 0 \tag{2.5}$$

shows that the first curvature of these surfaces vanishes. Accordingly the surfaces $\varphi(x^\alpha)$ = constant are minimal surfaces.

The condition

$$\underset{\sim}{n} = \psi \operatorname{grad} V \tag{2.25}$$

shows that the principal normal to vector-lines of $\underset{\sim}{s}$ is normal to the family of surfaces $V(x^\alpha)$ = constant. The vector-lines of $\underset{\sim}{s}$ are thus geodesics on the surfaces $V(x^\alpha)$ = constant.

It is seen that we have two sets of orthogonally intersecting families of surfaces, the set $\varphi(x^\alpha)$ = constant being minimal surfaces, and the orthogonal trajectories of the surfaces $\varphi(x^\alpha)$ = constant being geodesics on the family $V(x^\alpha)$ = constant. The families $\varphi(x^\alpha)$ = constant and $V(x^\alpha)$ = constant intersect on the vector-lines of $\underset{\sim}{b}$.

It is further noted that the Gaussian curvature of the minimal surfaces is given by $\tau_{\bar{s}}^2$ where

$$\tau_{\bar{s}}^2 = \tau_s^2 + \theta_{bs}^2 . \tag{2.34}$$

The asymptotic lines on a minimal surface are orthogonal. They are inclined at an angle ξ to the direction $-\underset{\sim}{n}$, where

$$\tan 2\xi = \frac{\theta_{bs}}{\tau_s} . \tag{2.35}$$

The geodesic curvatures of the n-lines and b-lines on the minimal surfaces $\varphi(x^\alpha)$ = constant are respectively div $\underset{\sim}{b}$ and $\varkappa_s$ + div $\underset{\sim}{n}$, while θ_{bs} and $-\theta_{bs}$ are the normal curvature of the n-lines and b-lines on the surfaces $\varphi(x^\alpha)$ = constant. The parameter θ_{bs} is also the geodesic curvature of the $\underset{\sim}{b}$-lines on the surface $V(x^\alpha)$ = constant. The geometrical configuration is shown in Figure 1.

From (2.19) and (2.25)

$$\text{curl}\ \underset{\sim}{n} = -\ \text{div}\ \underset{\sim\sim}{bs} - \theta_{bs}\ \underset{\sim}{b} , \tag{2.36}$$

and it follows from (2.25) and (2.36) that

$$\theta_{bs} = -\frac{\delta}{\delta s} \log \psi , \tag{2.37}$$

$$\text{div}\ \underset{\sim}{b} = \frac{\delta}{\delta b} \log \psi . \tag{2.38}$$

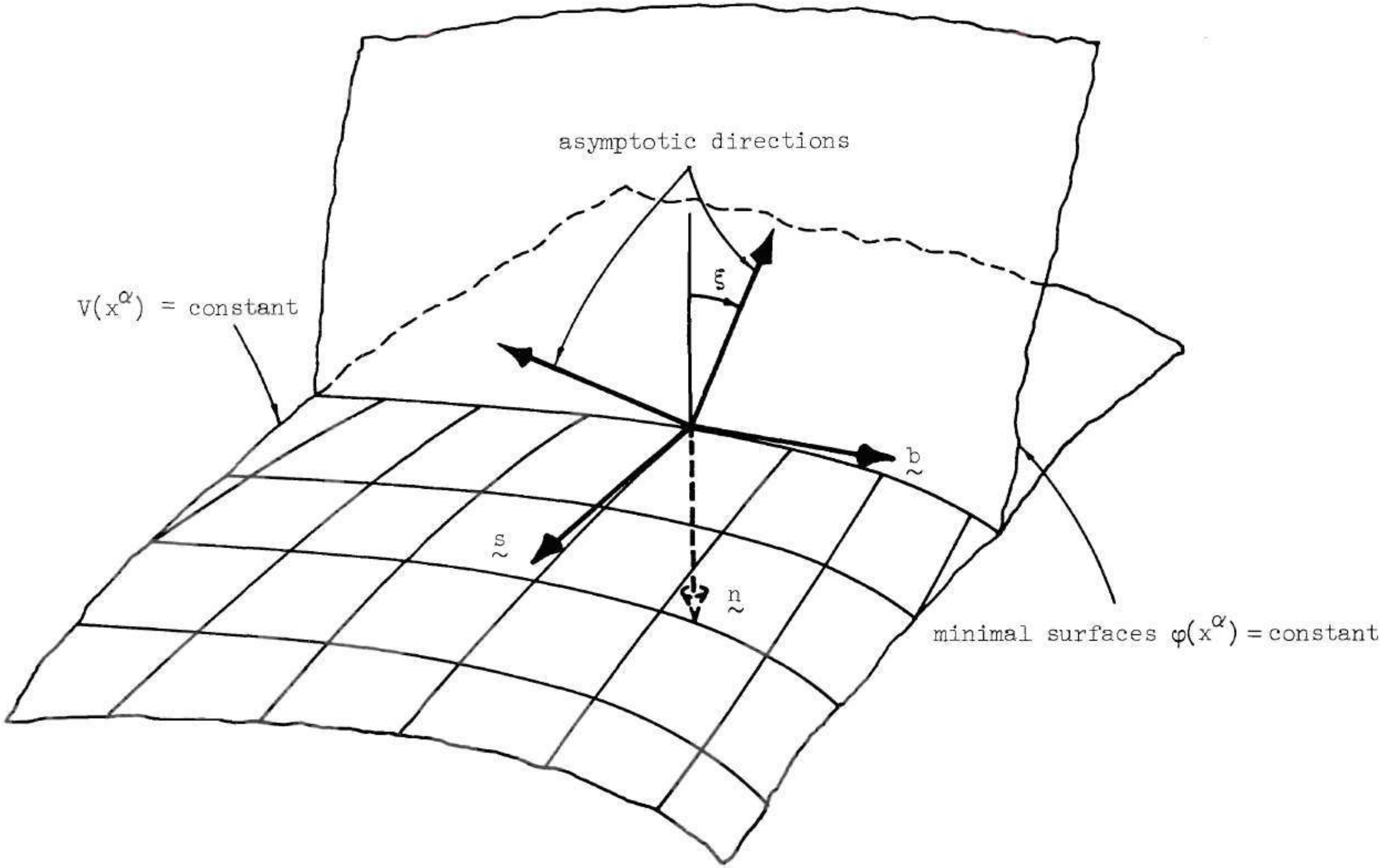

Figure 1. The Vector Field and its Associated Families of Surfaces.

The direction defined by curl $\underset{\sim}{n}$ being perpendicular to the surface gradient of the function ψ representing the distance between consecutive surfaces of the family $V(x^{\alpha})$ = constant is called the line of equidistance for the family.

From (2.32) and (2.33) and the conditions (2.37) and (2.38) one sees that the product $\psi\varkappa_s$ is constant over the representative surfaces $V(x^{\alpha})$ = constant. As the curvature $\varkappa_s$ of the vector lines of $\underset{\sim}{s}$ increases, so the surfaces come together.

Applying the formulae (2.29), (2.30), and (2.31) to the unit vectors $\underset{\sim}{s}$, $\underset{\sim}{n}$, and $\underset{\sim}{b}$, one obtains the nine compatibility conditions,* equivalent to the Gauss and Mainardi-Codazzi equations for the surfaces orthogonal to the s-lines and n-lines, and to the conditions div curl $\underset{\sim}{s} = 0$, div curl $\underset{\sim}{n} = 0$, and div curl $\underset{\sim}{b} = 0$ as follows:

$$\frac{\delta\theta_{bs}}{\delta s} + \theta_{bs}{}^2 - \varkappa_s(\varkappa_s + \text{div}\,\underset{\sim}{n}) - \tau_s{}^2 = 0 \,, \tag{2.39}$$

$$\frac{\delta\tau_s}{\delta s} - \varkappa_s\,\text{div}\,\underset{\sim}{b} + 2\theta_{bs}\,\tau_s = 0 \,, \tag{2.40}$$

$$\frac{\delta\varkappa_s}{\delta b} + \varkappa_s\,\text{div}\,\underset{\sim}{b} = 0 \,, \tag{2.41}$$

$$\frac{\delta\theta_{bs}}{\delta n} + \frac{\delta\tau_s}{\delta b} + 2\tau_s\,\text{div}\,\underset{\sim}{b} + 2\theta_{bs}(\varkappa_s + \text{div}\,\underset{\sim}{n}) = 0 \,, \tag{2.42}$$

*Special cases of these relations, corresponding to $\Omega_n = 0$, were derived by a different method in (1970 [1]), and general cases of these relations were shown in (1970 [3]). A more general expression of the relations equivalent to those in (1970 [3]) were derived in (1970 [2]).

$$\frac{\delta\tau_s}{\delta n} - \frac{\delta\theta_{bs}}{\delta b} + 2\tau_s(\varkappa_s + \operatorname{div} \underset{\sim}{n}) - 2\theta_{bs} \operatorname{div} \underset{\sim}{b} = 0 \;, \tag{2.43}$$

$$\frac{\delta\varkappa_s}{\delta n} - \varkappa_s{}^2 - 2(\tau_s{}^2 + \theta_{bs}{}^2) + \varkappa_s(\varkappa_s + \operatorname{div} \underset{\sim}{n}) = 0 \;, \tag{2.44}$$

$$\frac{\delta\theta_{bs}}{\delta b} + \frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{b} + \theta_{bs} \operatorname{div} \underset{\sim}{b} = 0 \;, \tag{2.45}$$

$$\frac{\delta\tau_s}{\delta b} - \frac{\delta}{\delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) - \theta_{bs}(2\varkappa_s + \operatorname{div} \underset{\sim}{n}) = 0 \;, \tag{2.46}$$

$$\frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} + \frac{\delta}{\delta n} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) + (\operatorname{div} \underset{\sim}{b})^2 + (\varkappa + \operatorname{div} \underset{\sim}{n})^2$$
$$- \tau_s{}^2 - \theta_{bs}{}^2 = 0 \;. \tag{2.47}$$

Eliminating $\frac{\delta\tau_s}{\delta b}$ from (2.42) and (2.46) by subtraction, one obtains

$$\frac{\delta}{\delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) + \frac{\delta\theta_{bs}}{\delta n} + \theta_{bs}(4\varkappa_s + 3 \operatorname{div} \underset{\sim}{n}) + 2\tau_s \operatorname{div} \underset{\sim}{b} = 0, \tag{2.48}$$

a relation, which, with

$$\frac{\delta\varkappa_s}{\delta s} = \theta_{bs}\, \varkappa_s \;, \tag{2.33}$$

yields

$$\frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{n} + \frac{\delta\theta_{bs}}{\delta n} + \theta_{bs}(5\varkappa_s + 3 \operatorname{div} \underset{\sim}{n}) + 2\tau_s \operatorname{div} \underset{\sim}{b} = 0 \;. \tag{2.49}$$

Again from (2.44), one obtains

$$\frac{\delta^2 \varkappa_s}{\delta s \delta n} = 4\theta_{bs} \frac{\delta \theta_{bs}}{\delta s} + 4\tau_s \frac{\delta \tau_s}{\delta s} - \frac{\delta \varkappa_s}{\delta s} \operatorname{div} \underset{\sim}{n} - \varkappa_s \frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{n} , \tag{2.50}$$

while from (2.33), there follows

$$\frac{\delta^2 \varkappa_s}{\delta n \delta s} = \varkappa_s \frac{\delta \theta_{bs}}{\delta n} + \theta_{bs} \frac{\delta \varkappa_s}{\delta n} . \tag{2.51}$$

The commutation formula (2.31) applied to $\varkappa_s$ gives

$$\frac{\delta^2 \varkappa_s}{\delta n \delta s} - \frac{\delta^2 \varkappa_s}{\delta s \delta n} = \varkappa_s \frac{\delta \varkappa_s}{\delta s} - \theta_{bs} \frac{\delta \varkappa_s}{\delta n} - 2\tau_s \frac{\delta \varkappa_s}{\delta b} . \tag{2.52}$$

From (2.50), (2.51), and (2.52), on substituting for $\frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{n} + \frac{\delta \theta_{bs}}{\delta n}$ the expression obtained from (2.44), and substituting for $\frac{\delta \theta_{bs}}{\delta s}$, $\frac{\delta \tau_s}{\delta s}$, $\frac{\delta \varkappa_s}{\delta s}$, $\frac{\delta \varkappa_s}{\delta n}$, and $\frac{\delta \varkappa_s}{\delta b}$, the expressions (2.39), (2.40), (2.33), (2.44), and (2.41), respectively, one obtains the relation

$$4\theta_{bs}(\tau_s^2 + \theta_{bs}^2) - \theta_{bs} \varkappa_s(5\varkappa_s + 4 \operatorname{div} \underset{\sim}{n}) - 4\tau_s \varkappa_s \operatorname{div} \underset{\sim}{b} = 0 . \tag{2.53}$$

By (2.33), (2.34), (2.39), (2.40), and (2.53)

$$\frac{\delta}{\delta s} [4(\tau_s^2 + \theta_{bs}^2) + \varkappa_s^2] = 0 , \tag{2.54}$$

or

$$\frac{\delta \tau_s^2}{\delta s} = - \frac{\theta_{bs}^2 \varkappa_s^2}{2} . \tag{2.55}$$

The relations (2.53), (2.54), and (2.55) were given by Hamel and represented the terminal point of his analysis in terms of vector-field geometry.

CHAPTER III

PRELIMINARY DEDUCTIONS

The purpose of this chapter is to establish five lemmas. Utilizing these lemmas, we discount, by reference, exceptional conditions arising in the main analysis.

Lemma 3.1.

In vector fields of the class defined in the statement of Hamel's theorem, the vanishing of the curvature κ_s in a neighborhood implies the vanishing of θ_{bs}.

When κ_s is zero, the vector magnitude V is spatially constant. As it has been noted, however, the formalism of the preceding chapter remains valid in this case with $\underset{\sim}{n}$ defined by (2.11). In particular

$$\operatorname{grad} \underset{\sim}{s} = \begin{matrix} -\theta_{bs}\,\underset{\sim\sim}{nn} - \tau_s\,\underset{\sim\sim}{nb} \\ \\ -\tau_s\,\underset{\sim\sim}{bn} + \theta_{bs}\,\underset{\sim\sim}{bb} \end{matrix} \quad . \tag{3.1}$$

One may deduce the condition (2.44) as before; it takes the form

$$\tau_s^{\ 2} + \theta_{bs}^{\ 2} = 0 \ . \tag{3.2}$$

It follows from (3.2) that τ_s and θ_{bs} must each vanish. This establishes Lemma 3.1.

Lemma 3.2.

Vector fields of the class defined in the statement of Hamel's theorem, and for which θ_{bs} vanishes in a neighborhood, consist of parallel straight lines, concentric circles, or circular helices mounted on concentric circular cylinders.

With vanishing of θ_{bs}, (2.53) reduces to

$$\tau_s \, \varkappa_s \, \operatorname{div} \underset{\sim}{b} = 0 \ , \tag{3.3}$$

so that one or other of τ_s, $\varkappa_s$, or div $\underset{\sim}{b}$ must be zero.

If $\varkappa_s$ vanishes in a neighborhood, it follows from (2.44) that τ_s must also vanish. Thus grad $\underset{\sim}{s}$ is zero. The vector field must consist of parallel straight lines.

If τ_s vanishes the vector lines are plane curves. Also by (2.40)

$$\varkappa_s \operatorname{div} \underset{\sim}{b} = 0 \ , \tag{3.4}$$

so that either $\varkappa_s$ or div $\underset{\sim}{b}$ must vanish. If $\varkappa_s$ vanishes, it follows from Lemma 3.1 that the vector field consists of parallel straight lines. If div $\underset{\sim}{b}$ vanishes and $\varkappa_s$ does not vanish, then (2.33) shows that $\varkappa_s$ must be constant along the vector line. The vector-lines, being plane curves, must be concentric circles.

The remaining possibility is that div $\underset{\sim}{b}$ and θ_{bs} each vanish while τ_s and $\varkappa_s$ do not vanish. In this case the conditions (2.32), (2.33), (2.40), and (2.41) show that the curvature $\varkappa_s$ and the torsion τ_s must each be constant over the surface $V(x^\alpha)$ = constant.

It follows that the vector field must consist of circular helices mounted on concentric circular cylinders, the cylinders being the surfaces on which the vector magnitude maintains a constant value. This establishes Lemma 3.2.

It should be emphasized here that Hamel's theorem will follow from Lemma 3.2. if it can be asserted that no real fields of the class defined in the statement of Hamel's theorem are possible when θ_{bs} is different from zero in a neighborhood.

One may now show that the vanishing of div $\underset{\sim}{b}$ is equivalent to the vanishing of θ_{bs}. Thus Hamel's theorem also follows if it can be shown that div $\underset{\sim}{b} = 0$. We have:

Lemma 3.3.

In vector fields of the class defined in the statement of Hamel's theorem, if div $\underset{\sim}{b}$ vanishes in a neighborhood, or equivalently if the unit bi-normal vector is solenoidal, then the vector fields are either circular helical or consist of parallel straight lines.

To prove this lemma, it is first noted that when div $\underset{\sim}{b}$ vanishes in a neighborhood, the condition (2.53) requires that either θ_{bs} is zero, in which case the result follows from Lemma 3.2, or else

$$\theta_{bs}^{2} + \tau_{s}^{2} - \frac{\varkappa_{s}^{2}}{4} - \varkappa_{s}(\varkappa_{s} + \mathrm{div}\ \underset{\sim}{n}) = 0\ . \tag{3.6}$$

It will be shown that (3.6) is impossible in a neighborhood. Taking the $\underset{\sim}{s}$-gradient of (3.6) and using (2.33) and (2.54) one has, for non-vanishing $\varkappa_s$,

$$\frac{\delta}{\delta s}\ (\varkappa_{s} + \mathrm{div}\ \underset{\sim}{n}) = -\ \theta_{bs}(2\varkappa_{s} + \mathrm{div}\ \underset{\sim}{n}) \tag{3.7}$$

so that, by (2.46)

$$\frac{\delta \tau_s}{\delta b} = 0 \ . \tag{3.8}$$

By (2.30), and (3.8),

$$\frac{\delta^2 \tau_s}{\delta b \delta s} = 0 \ , \tag{3.9}$$

so that, by (2.40)

$$\frac{\delta}{\delta b} (\theta_{bs} \ \tau_s) = 0 \ , \tag{3.10}$$

and hence, by (3.8) and (3.10)

$$\frac{\delta \theta_{bs}}{\delta b} = 0 \ . \tag{3.11}$$

Multiplying (2.42) and (2.43) by θ_{bs} and τ_s respectively, and adding, one obtains from (3.8) and (3.11),

$$\frac{\delta}{\delta n} (\theta_{bs}{}^2 + \tau_s{}^2) = - 4(\theta_{bs}{}^2 + \tau_s{}^2)(\varkappa_s + \operatorname{div} \underset{\sim}{n}) \tag{3.12}$$

Taking the $\underset{\sim}{n}$-gradient of (3.6), and substituting for $\frac{\delta \varkappa_s}{\delta n}$ from (2.44), for $\frac{\delta}{\delta n} (\varkappa_s + \operatorname{div} \underset{\sim}{n})$ from (2.47) and $\frac{\delta}{\delta n} (\theta_{bs}{}^2 + \tau_s{}^2)$ from (3.12), one obtains, after a little reduction, for $\varkappa_s$ nonvanishing,

$$\operatorname{div} \underset{\sim}{n} = - \frac{3}{2} \varkappa_s \ . \tag{3.13}$$

Substituting (3.13) into (3.6) one finally obtains

$$4(\theta_{bs}^{\;2} + \tau_s^{\;2}) + \varkappa_s^{\;2} = 0 \;, \tag{3.14}$$

showing that θ_{bs}, τ_s, and $\varkappa_s$ must each be zero. This establishes the lemma.

A review of the analysis used to prove Lemma 3.3 shows that it can be carried through in an analagous, but somewhat simpler, manner if τ_s is zero while $\varkappa_s$ is non-vanishing.

One may also note, by (2.40), that the vanishing of τ_s in a neighborhood implies

$$\varkappa_s \operatorname{div} \underset{\sim}{b} = 0 \;, \tag{3.15}$$

so that for non-vanishing $\varkappa_s$, div $\underset{\sim}{b}$ must be zero. Hence by Lemma 3.3, θ_{bs} must be zero.

From these observations one deduces Lemma 3.4.

In vector fields of the class defined in the statement of Hamel's theorem, the vanishing of torsion τ_s in a neighborhood implies the vanishing of θ_{bs}. In this case the vector-lines of $\underset{\sim}{s}$ are either concentric circles or parallel straight lines.

The final lemma established in this section is required for the elimination of possible exceptions due to the vanishing of a factor occurring in the analysis of the next chapter.

Lemma 3.5

The vanishing of the expression

$$D = \frac{\tau_{\bar{s}}^{\;2} \varkappa_s}{\theta_{bs}} \operatorname{div} \underset{\sim}{b} + \tau_s\left(\tau_{\bar{s}}^{\;2} + \frac{\varkappa_s^{\;2}}{4}\right) \;, \tag{3.16}$$

where $\tau_{\bar{s}}^2 = \theta_{bs}^2 + \tau_s^2$, gives no exception to Hamel's theorem.

If D = 0, one has

$$\operatorname{div} \underset{\sim}{b} = - \frac{\theta_{bs}\tau_s}{\tau_{\bar{s}}^2 \varkappa_s} \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right), \tag{3.17}$$

where neither $\tau_{\bar{s}}^2$ nor $\varkappa_s$ vanishes because to attain non-vanishing of θ_{bs} one requires, by virtue of Lemma 3.1 and Lemma 3.4, that none of τ_s or $\varkappa_s$ vanish. Therefore, according to the definition, $\tau_{\bar{s}}^2$ will also not vanish.

Substituting (3.17) into (2.40),

$$\frac{\delta\varkappa_s}{\delta b} = \frac{\tau_s\theta_{bs}}{\tau_{\bar{s}}^2} \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right). \tag{3.18}$$

Taking the $\underset{\sim}{s}$-gradient of (3.18) and using (2.54),

$$\frac{\delta^2\varkappa_s}{\delta s\,\delta b} = \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right)\left(\frac{\theta_{bs}}{\tau_{\bar{s}}^2}\frac{\delta\tau_s}{\delta s} + \frac{\tau_s}{\tau_{\bar{s}}^2}\frac{\delta\theta_{bs}}{\delta s} - \frac{\tau_s\theta_{bs}}{\tau_{\bar{s}}^4}\frac{\delta\tau_{\bar{s}}^2}{\delta s}\right). \tag{3.19}$$

Substituting (3.17) into (2.40),

$$\frac{\delta\tau_s}{\delta s} = - \frac{\tau_s\theta_{bs}}{\tau_{\bar{s}}^2} \left(3\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right). \tag{3.20}$$

From (2.53),

$$\varkappa_s + \operatorname{div} \underset{\sim}{n} = - \frac{\tau_s}{\theta_{bs}} \operatorname{div} \underset{\sim}{b} + \frac{1}{\varkappa_s}\left(\tau_{\bar{s}}^2 - \frac{\varkappa_s^2}{4}\right), \text{ since } \theta_{bs} \neq 0,\ \varkappa_s \neq 0, \tag{3.21}$$

on substituting for div $\underset{\sim}{b}$, the expression (3.17), (3.21) becomes

$$\varkappa_s + \operatorname{div} \underset{\sim}{n} = \frac{1}{\tau_{\bar{s}}^{2} \varkappa_s} \left[\tau_{\bar{s}}^{4} + \left(\tau_s^{2} - \frac{\varkappa_s^{2}}{4}\right)\tau_{\bar{s}}^{2} + \frac{\tau_s^{2} \varkappa_s^{2}}{4}\right] . \tag{3.22}$$

Substituting (3.22) into (2.39),

$$\frac{\delta\theta_{bs}}{\delta s} = \frac{1}{\tau_{\bar{s}}^{2}} \left(3\tau_s^{2}\tau_{\bar{s}}^{2} - \frac{\theta_{bs}^{2}\varkappa_s^{2}}{4}\right) . \tag{3.23}$$

From (3.19), on substituting for $\frac{\delta\tau_s}{\delta s}$, $\frac{\delta\theta_{bs}}{\delta s}$, and $\frac{\delta\tau_s^{2}}{\delta s}$, the expressions (3.20), (3.23), and (2.55), respectively, one has

$$\frac{\delta^2\varkappa_s}{\delta s\,\delta b} = -\frac{3\tau_2}{\tau_{\bar{s}}^{2}} \left(\tau_{\bar{s}}^{2} + \frac{\varkappa_s^{2}}{4}\right)\left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right) . \tag{3.24}$$

Taking the $\underset{\sim}{b}$-gradient of (2.33),

$$\begin{aligned} \frac{\delta^2\varkappa_s}{\delta b\,\delta s} &= \varkappa_s \frac{\delta\theta_{bs}}{\delta b} + \theta_{bs} \frac{\delta\varkappa_s}{\delta b} \\ &= \varkappa_s \frac{\delta\theta_{bs}}{\delta b} + \frac{\tau_s\theta_{bs}^{2}}{\tau_{\bar{s}}^{2}} \left(\tau_{\bar{s}}^{2} + \frac{\varkappa_s^{2}}{4}\right) , \quad \text{by (3.18)} . \end{aligned} \tag{3.25}$$

Applying the commutation formula (2.30) to $\varkappa_s$,

$$\frac{\delta^2\varkappa_s}{\delta s\,\delta b} - \frac{\delta^2\varkappa_s}{\delta b\,\delta s} = -\,\theta_{bs} \frac{\delta\varkappa_s}{\delta b}$$

$$= -\frac{\tau_s \theta_{bs}^{\;2}}{\tau_{\bar{s}}^{\;2}} \left(\tau_{\bar{s}}^{\;2} + \frac{\varkappa_s^{\;2}}{4}\right), \quad \text{by (3.18)} . \tag{3.26}$$

From (3.26), on substituting for $\frac{\delta^2 \varkappa_s}{\delta s \delta b}$, $\frac{\delta^2 \varkappa_s}{\delta b \delta s}$ the expressions (3.24) and (3.25), respectively, one obtains

$$\frac{\delta \theta_{bs}}{\delta b} = -\frac{3\tau_s}{\tau_{\bar{s}}^{\;2} \varkappa_s} \left(\tau_{\bar{s}}^{\;2} + \frac{\varkappa_s^{\;2}}{4}\right)\left(\tau_{\bar{s}}^{\;2} - 2\tau_s^{\;2}\right) . \tag{3.27}$$

From (2.44), on substituting for $\varkappa_s + \operatorname{div} \underset{\sim}{n}$, the expression (3.22), one has

$$\frac{\delta \varkappa_s}{\delta n} = \frac{1}{\tau_{\bar{s}}^{\;2}} \left[\tau_{\bar{s}}^{\;4} - \left(\tau_s^{\;2} - \tfrac{5}{4}\varkappa_s^{\;2}\right)\tau_{\bar{s}}^{\;2} - \frac{\tau_s^{\;2} \varkappa_s^{\;2}}{4}\right] . \tag{3.28}$$

Taking the $\underset{\sim}{s}$-gradient of (3.28),

$$\frac{\delta^2 \varkappa_s}{\delta s \delta n} = \frac{\varkappa_s}{2\tau_{\bar{s}}^{\;2}} \left(5\tau_{\bar{s}}^{\;2} - \tau_s^{\;2}\right) \frac{\delta \varkappa_s}{\delta s}$$

$$+ \frac{1}{\tau_{\bar{s}}^{\;4}} \left(\tau_{\bar{s}}^{\;4} + \frac{\tau_s^{\;2} \varkappa_s^{\;2}}{4}\right) \frac{\delta \tau_{\bar{s}}^{\;2}}{\delta s} - \frac{2\tau_s}{\tau_{\bar{s}}^{\;2}} \left(\tau_{\bar{s}}^{\;2} + \frac{\varkappa_s^{\;2}}{4}\right) \frac{\delta \tau_s}{\delta s} . \tag{3.29}$$

From (3.29), on substituting for $\frac{\delta \varkappa_s}{\delta s}$, $\frac{\delta \tau_{\bar{s}}^{\;2}}{\delta s}$, and $\frac{\delta \tau_s}{\delta s}$, the expressions (2.33), (2.55), and (3.20), respectively, one has

$$\frac{\delta^2 \varkappa_s}{\delta s \delta n} = \frac{\theta_{bs}}{\tau_{\bar{s}}^{\;2}} \left[\left(6\tau_s^{\;2} + 2\varkappa_s^{\;2}\right)\tau_{\bar{s}}^{\;2} + \frac{3}{2}\, \tau_s^{\;2} \varkappa_s^{\;2}\right] . \tag{3.30}$$

Taking the $\underset{\sim}{n}$-gradient of (2.33),

$$\frac{\delta^2 \varkappa_s}{\delta n\,\delta s} = \varkappa_s \frac{\delta \theta_{bs}}{\delta n} + \theta_{bs} \frac{\delta \varkappa_s}{\delta n}$$

$$= \varkappa_s \frac{\delta \theta_{bs}}{\delta n} + \frac{\theta_{bs}}{\tau_{\bar{s}}^2}\left[\tau_{\bar{s}}^4 - \left(\tau_s^2 - \frac{5}{4}\varkappa_s^2\right)\tau_{\bar{s}}^2 - \frac{\tau_s^2 \varkappa_s^2}{4}\right],$$

$$\text{by (3.28)}\ . \qquad (3.31)$$

Applying the commutation formula (2.31) to $\varkappa_s$,

$$\frac{\delta^2 \varkappa_s}{\delta n\,\delta s} - \frac{\delta^2 \varkappa_s}{\delta s\,\delta n} = \varkappa_s \frac{\delta \varkappa_s}{\delta s} - \theta_{bs} \frac{\delta \varkappa_s}{\delta n} - 2\tau_s \frac{\delta \varkappa_s}{\delta b}\ . \qquad (3.32)$$

From (2.32), on substituting for $\frac{\delta^2 \varkappa_s}{\delta n\,\delta s}$, $\frac{\delta^2 \varkappa_s}{\delta s\,\delta n}$, $\frac{\delta \varkappa_s}{\delta s}$, $\frac{\delta \varkappa_s}{\delta n}$, and $\frac{\delta \varkappa_s}{\delta b}$, the expressions (3.31), (3.30), (2.33), (3.28) and (3.18), respectively, one obtains

$$\frac{\delta \theta_{bs}}{\delta n} = -\frac{\theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s}\left[2\tau_{\bar{s}}^4 - \left(6\tau_s^2 + \frac{\varkappa_s^2}{2}\right)\tau_{\bar{s}}^2 - \frac{3}{2}\tau_s^2 \varkappa_s^2\right]. \qquad (3.33)$$

Taking the $\underset{\sim}{s}$-gradient of (3.27) and using (2.54),

$$\frac{\delta^2 \theta_{bs}}{\delta s\,\delta b} = 3\left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right)\left[\frac{\tau_s}{\tau_{\bar{s}}^2 \varkappa_s^2}\left(\tau_{\bar{s}}^2 - 2\tau_s^2\right)\frac{\delta \varkappa_s}{\delta s}\right.$$

$$\left. - \frac{2\tau_s^3}{\tau_{\bar{s}}^4 \varkappa_s}\frac{\delta \tau_{\bar{s}}^2}{\delta s} - \frac{1}{\tau_{\bar{s}}^2 \varkappa_s}\left(\tau_{\bar{s}}^2 - 6\tau_s^2\right)\frac{\delta \tau_s}{\delta s}\right]. \qquad (3.34)$$

From (3.34), on substituting for $\frac{\delta \varkappa_s}{\delta s}$, $\frac{\delta \tau_{\bar{s}}^2}{\delta s}$, and $\frac{\delta \tau_s}{\delta s}$, the expressions (2.33), (2.55), and (3.20), respectively, one has

$$\frac{\delta^2 \theta_{bs}}{\delta s \delta b} = \frac{3\tau_s \theta_{bs}}{\tau_{\bar{s}}^4 \varkappa_s} \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right)\left[4\tau_{\bar{s}}^4 - \left(20\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\tau_{\bar{s}}^2 - \frac{\tau_s^2 \varkappa_s^2}{2}\right] \qquad (3.35)$$

Taking the $\underset{\sim}{b}$-gradient of (3.23) and substituting for $\frac{\delta \tau_{\bar{s}}^2}{\delta b}$, the expression obtained by taking the $\underset{\sim}{b}$-gradient of (2.34),

$$\frac{\delta^2 \theta_{bs}}{\delta b \delta s} = \frac{\tau_s}{\tau_{\bar{s}}^4} \left(6\tau_{\bar{s}}^4 + \frac{\theta_{bs}^2 \varkappa_s^2}{2}\right)\frac{\delta \tau_s}{\delta b}$$

$$- \frac{\tau_s^2 \theta_{bs} \varkappa_s^2}{2\tau_{\bar{s}}^4} \frac{\delta \theta_{bs}}{\delta b} - \frac{\theta_{bs}^2 \varkappa_s}{2\tau_{\bar{s}}^2} \frac{\delta \varkappa_s}{\delta b} . \qquad (3.36)$$

From (2.42), on substituting for $\frac{\delta \theta_{bs}}{\delta n}$ and $\varkappa_s + \operatorname{div} \underset{\sim}{n}$, the expressions (3.33) and (3.22), respectively, one obtains

$$\frac{\delta \tau_s}{\delta b} = - \frac{6\tau_s^2 \theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s} \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right) . \qquad (3.37)$$

From (3.36), on substituting for $\frac{\delta \tau_s}{\delta b}$, $\frac{\delta \theta_{bs}}{\delta b}$, and $\frac{\delta \varkappa_s}{\delta b}$, the expressions (3.37), (3.27), and (3.18), respectively, one has

$$\frac{\delta^2 \theta_{bs}}{\delta b \delta s} = - \frac{\tau_s \theta_{bs}}{\tau_{\bar{s}}^4 \varkappa_s} \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right)\left[\left(36\tau_s^2 + \frac{\varkappa_s^2}{2}\right)\tau_{\bar{s}}^2 + \tau_s^2 \varkappa_s^2\right] . \qquad (3.38)$$

Applying the commutation formula (2.30) to θ_{bs},

$$\frac{\delta^2 \theta_{bs}}{\delta s \delta b} - \frac{\delta^2 \theta_{bs}}{\delta b \delta s} = - \theta_{bs} \frac{\delta \theta_{bs}}{\delta b}$$

$$= \frac{3 \tau_s \theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s} \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right)\left(\tau_{\bar{s}}^2 - 2\tau_s^2\right) \text{ , by (2.27)}. \quad (3.39)$$

From (3.39), on substituting for $\frac{\delta^2 \theta_{bs}}{\delta s \delta b}$ and $\frac{\delta^2 \theta_{bs}}{\delta b \delta s}$, the expressions (3.35) and (3.38), respectively, one obtains

$$\frac{\tau_s \theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s} \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right)\left[9\tau_{\bar{s}}^4 - \left(18\tau_s^2 - \frac{5}{4}\varkappa_s^2\right)\tau_{\bar{s}}^2 - \frac{\tau_s^2 \varkappa_s^2}{2}\right] = 0 \quad (3.40)$$

Since none of θ_{bs}, τ_s, or $\varkappa_s$ vanish, $\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}$ is different from zero. Therefore, (3.40) yields

$$9\tau_{\bar{s}}^4 - \left(18\tau_s^2 - \frac{5}{4}\varkappa_s^2\right)\tau_{\bar{s}}^2 - \frac{\tau_s^2 \varkappa_s^2}{2} = 0 \ . \quad (3.41)$$

Taking the $\underset{\sim}{s}$-gradient of (3.33),

$$\frac{\delta^2 \theta_{bs}}{\delta s \delta n} = \frac{\theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s^2} \left[2\tau_{\bar{s}}^4 - \left(6\tau_s^2 - \frac{\varkappa_s^2}{2}\right)\tau_{\bar{s}}^2 + \frac{3}{2}\tau_s^2 \varkappa_s^2\right] \frac{\delta \varkappa_s}{\delta s}$$

$$- \frac{\theta_{bs}}{\tau_{\bar{s}}^4 \varkappa_s} \left(2\tau_{\bar{s}}^4 + \frac{3}{2}\tau_s^2 \varkappa_s^2\right) \frac{\delta \tau_{\bar{s}}^2}{\delta s} + \frac{12 \tau_s \theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s} \left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right) \frac{\delta \tau_s}{\delta s}$$

$$- \frac{1}{\tau_{\bar{s}}^2 \varkappa_s} \left[2\tau_{\bar{s}}^4 - \left(6\tau_s^2 + \frac{\varkappa_s^2}{2}\right)\tau_{\bar{s}}^2 - \frac{3}{2}\tau_s^2 \varkappa_s^2\right] \frac{\delta \theta_{bs}}{\delta s} \ . \quad (3.42)$$

From (3.42), on substituting for $\frac{\delta \varkappa_s}{\delta s}$, $\frac{\delta \tau_s^{-2}}{\delta s}$, $\frac{\delta \tau_s}{\delta s}$, and $\frac{\delta \theta_{bs}}{\delta s}$, the expressions (2.33), (2.55), (3.20), and (3.23), respectively, one has

$$\frac{\delta^2 \theta_{bs}}{\delta s \delta n} = \frac{1}{\tau_s^{-4} \varkappa_s} \Big[2\tau_s^{-8} + (10\tau_s^{\,2} + 2\varkappa_s^{\,2})\tau_s^{-6} - 4\tau_s^{\,2}(15\theta_{bs}^{\,2} - \varkappa_s^{\,2})\tau_s^{-4}$$

$$- \frac{\theta_{bs}^{\,2}\varkappa_s^{\,2}}{2}\left(33\tau_s^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right)\tau_s^{-2} - \frac{3}{8}\,\tau_s^{\,2}\theta_{bs}^{\,2}\varkappa_s^{\,4}\Big]. \tag{3.43}$$

Taking the $\underset{\sim}{n}$-gradient of (3.23) and substituting for $\frac{\delta \tau_s^{-2}}{\delta n}$, the expression obtained by taking the $\underset{\sim}{n}$-gradient of (2.34),

$$\frac{\delta^2 \theta_{bs}}{\delta n \delta s} = \frac{\tau_s}{\tau_s^{-4}}\left(6\tau_s^{-4} + \frac{\theta_{bs}^{\,2}\varkappa_s^{\,2}}{2}\right)\frac{\delta \tau_s}{\delta n}$$

$$- \frac{\tau_s^{\,2}\theta_{bs}\varkappa_s^{\,2}}{2\tau_s^{-4}}\,\frac{\delta \theta_{bs}}{\delta n} - \frac{\theta_{bs}^{\,2}\varkappa_s}{2\tau_s^{-2}}\,\frac{\delta \varkappa_s}{\delta n}. \tag{3.44}$$

From (2.43), on substituting for $\frac{\delta \theta_{bs}}{\delta b}$, $\varkappa + \operatorname{div} \underset{\sim}{n}$ and $\operatorname{div} \underset{\sim}{b}$, the expressions (3.27), (3.22), and (3.17), respectively, one obtains

$$\frac{\delta \tau_s}{\delta n} = -\frac{\tau_s}{\tau_s^{-2}\varkappa_s}\left[7\tau_s^{-4} - \left(6\tau_s^{\,2} - \frac{3}{4}\,\varkappa_s^{\,2}\right)\tau_s^{-2} - \frac{3}{2}\,\tau_s^{\,2}\varkappa_s^{\,2}\right]. \tag{3.45}$$

From (3.44), on substituting for $\frac{\delta \tau_s}{\delta n}$, $\frac{\delta \theta_{bs}}{\delta n}$, and $\frac{\delta \varkappa_s}{\delta n}$, the expressions (3.45), (3.33), and (3.28), respectively, one has

$$\frac{\delta^2\theta_{bs}}{\delta n\delta s} = -\frac{1}{\tau_s^{-4}\varkappa_s}\Big[\Big(42\tau_s^2 + \frac{\varkappa_s^2}{2}\Big)\tau_s^{-6} - 6\tau_s^2(6\tau_s^2 - \varkappa_s^2)\tau_s^{-4}$$

$$- \varkappa_s^2\Big(11\tau_s^4 - \frac{5}{8}\theta_{bs}^2\varkappa_s^2\Big)\tau_s^{-2} + \frac{1}{2}\tau_s^2\theta_{bs}^2\varkappa_s^4\Big] . \qquad (3.46)$$

Applying the commutation formula (2.31) to θ_{bs}.

$$\frac{\delta^2\theta_{bs}}{\delta n\delta s} - \frac{\delta^2\theta_{bs}}{\delta s\delta n} = \varkappa_s\frac{\delta\theta_{bs}}{\delta s} - \theta_{bs}\frac{\delta\theta_{bs}}{\delta n} - 2\tau_s\frac{\delta\theta_{bs}}{\delta b} . \qquad (3.47)$$

From (3.47), on substituting for $\frac{\delta^2\theta_{bs}}{\delta n\delta s}$, $\frac{\delta^2\theta_{bs}}{\delta s\delta n}$, $\frac{\delta\theta_{bs}}{\delta s}$, $\frac{\delta\theta_{bs}}{\delta n}$, and $\frac{\delta\theta_{bs}}{\delta b}$, the expressions (3.46), (3.43), (3.23), (3.33), and (3.27), respectively, one obtains

$$\frac{1}{\tau_s^{-4}\varkappa_s}\Big[4\tau_s^{-8} + (8\tau_s^2 + 2\varkappa_s^2)\tau_s^{-6} - \tau_s^2(18\theta_{bs}^2 - \varkappa_s^2)\tau_s^{-4}$$

$$- \theta_{bs}^2\varkappa_s^2\Big(4\tau_s^2 - \frac{\varkappa_s^2}{4}\Big)\tau_s^{-2} + \frac{1}{8}\tau_s^2\theta_{bs}^2\varkappa_s^4\Big] = 0 , \quad (3.48)$$

and therefore,

$$4\tau_s^{-8} + (8\tau_s^2 + 2\varkappa_s^2)\tau_s^{-6} - \tau_s^2(18\theta_{bs}^2 - \varkappa_s^2)\tau_s^{-4}$$

$$- \theta_{bs}^2\varkappa_s^2\Big(4\tau_s^2 - \frac{\varkappa_s^2}{4}\Big)\tau_s^{-2} + \frac{1}{8}\tau_s^2\theta_{bs}^2\varkappa_s^4 = 0 . \qquad (3.49)$$

Multiplying (3.41) by $\tau_s^{\,2}\bar{\tau}_s^{\,2}$, then adding to (3.49) and reducing by the definition (2.34), one obtains the following sum of squares:

$$3\bar{\tau}_s^{\,8} + \left(\theta_{bs}^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right)\bar{\tau}_s^{\,6} + \frac{7}{4}\,\theta_{bs}^{\,2}\varkappa_s^{\,2}\bar{\tau}_s^{\,4}$$

$$+ \frac{\varkappa_s^{\,2}}{2}\left(7\tau_s^{\,4} + \frac{1}{2}\,\theta_{bs}^{\,2}\varkappa_s^{\,2}\right)\bar{\tau}_s^{\,2} + \frac{1}{8}\,\tau_s^{\,2}\theta_{bs}^{\,2}\varkappa_s^{\,4} = 0 \qquad (3.50)$$

We see that

$$\bar{\tau}_s^{\,2} = \theta_{bs}^{\,2} + \tau_s^{\,2} \qquad (2.34)$$

must vanish, and accordingly, θ_{bs} and τ_s must each be zero. The only possibilities are the concentric circular or rectilinear vector fields.

CHAPTER IV

DERIVATION OF POLYNOMIAL EXPRESSIONS FOR $\frac{\delta\theta_{bs}}{\delta n}$ AND $\frac{\delta\theta_{bs}}{\delta b}$

Since from Lemmas 3.1 to 3.4 one notes that exceptions to the rectilinear or circular helical configuration can occur only if each of the four parameters θ_{bs}, τ_s, $\varkappa_s$, and div $\underset{\sim}{b}$ is non-vanishing, Hamel's condition (2.53) can be written in the following two forms:

$$\varkappa_s + \text{div } n = - \frac{\tau_s}{\theta_{bs}} \text{div } b + \frac{1}{\varkappa_s}\left(\tau_s^2 - \frac{\varkappa_s^2}{4}\right), \qquad (4.1)$$

or

$$\text{div } \underset{\sim}{b} = - \frac{\theta_{bs}}{\tau_s}\left[(\varkappa_s + \text{div } \underset{\sim}{n}) - \frac{1}{\varkappa_s}\left(\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\right] . \qquad (4.2)$$

Taking the $\underset{\sim}{n}$-gradient of (4.1),

$$\frac{\delta}{\delta n}(\varkappa_s + \text{div } \underset{\sim}{n}) = - \frac{\tau_s}{\theta_{bs}} \frac{\delta}{\delta n} \text{div } \underset{\sim}{b} + \left(\frac{\tau_s}{\theta_{bs}^2} \text{div } \underset{\sim}{b} + \frac{2\theta_{bs}}{\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta n}$$

$$- \left(\frac{\text{div } \underset{\sim}{b}}{\theta_{bs}} - \frac{2\tau_s}{\varkappa_s}\right)\frac{\delta\tau_s}{\delta n} - \frac{1}{\varkappa_s^2}\left(\tau_s^2 + \frac{\varkappa_s^2}{4}\right)\frac{\delta\varkappa_s}{\delta n} . \qquad (4.3)$$

From (4.3), on substituting for $\frac{\delta\tau_s}{\delta n}$ and $\frac{\delta\varkappa_s}{\delta n}$, the expressions obtained by substituting (4.1) into (2.43) and (2.44), respectively, one obtains

$$\frac{\delta}{\delta n}\left(\varkappa_s + \operatorname{div} \underset{\sim}{n}\right) = -\frac{\tau_s}{\theta_{bs}}\frac{\delta}{\delta n}\operatorname{div} \underset{\sim}{b} + \left(\frac{\tau_s}{\theta_{bs}^{\;2}}\operatorname{div} \underset{\sim}{b} + \frac{2\theta_{bs}}{\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta n}$$

$$-\left(\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} - \frac{2\tau_s}{\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta b} - \frac{2\bar{\tau}_s^{\;2}}{\theta_{bs}^{\;2}}\left(\operatorname{div} \underset{\sim}{b}\right)^2$$

$$+\frac{\tau_s}{\theta_{bs}\varkappa_s}\left(5\bar{\tau}_s^{\;2} - \frac{3}{4}\varkappa_s^{\;2}\right)\operatorname{div} \underset{\sim}{b}$$

$$-\frac{1}{\varkappa_s^{\;2}}\left[\bar{\tau}_s^{\;4} + \left(4\tau_s^{\;2} + \frac{3}{2}\varkappa_s^{\;2}\right)\bar{\tau}_s^{\;2} - \varkappa_s^{\;2}\left(\tau_s^{\;2} - \frac{5}{16}\varkappa_s^{\;2}\right)\right] \quad (4.4)$$

Taking the $\underset{\sim}{b}$-gradient of (4.2),

$$\frac{\delta}{\delta b}\operatorname{div} \underset{\sim}{b} = -\frac{\theta_{bs}}{\tau_s}\frac{\delta}{\delta b}\left(\varkappa_s + \operatorname{div} \underset{\sim}{n}\right) - \left\{\frac{1}{\tau_s}\left[\left(\varkappa_s + \operatorname{div} \underset{\sim}{n}\right) - \frac{1}{\varkappa_s}\left(\bar{\tau}_s^{\;2} - \frac{\varkappa_s^{\;2}}{4}\right)\right]\right.$$

$$\left.-\frac{2\theta_{bs}^{\;2}}{\tau_s\varkappa_s}\right\}\frac{\delta\theta_{bs}}{\delta b} + \frac{\theta_{bs}}{\tau_s}\left\{\frac{1}{\tau_s}\left[\left(\varkappa_s + \operatorname{div} \underset{\sim}{n}\right) - \frac{1}{\varkappa_s}\left(\bar{\tau}_s^{\;2} - \frac{\varkappa_s^{\;2}}{4}\right)\right]\right.$$

$$\left.+\frac{2\tau_s}{\varkappa_s}\right\}\frac{\delta\tau_s}{\delta b} - \frac{\theta_{bs}}{\tau_s\varkappa_s^{\;2}}\left(\bar{\tau}_s^{\;2} + \frac{\varkappa_s^{\;2}}{4}\right)\frac{\delta\varkappa_s}{\delta b}$$

$$= -\frac{\theta_{bs}}{\tau_s}\frac{\delta}{\delta b}\left(\varkappa_s + \operatorname{div} \underset{\sim}{n}\right) + \left(\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} + \frac{2\theta_{bs}^{\;2}}{\tau_s\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta b}$$

$$-\left(\frac{\operatorname{div} \underset{\sim}{b}}{\tau_s} - \frac{2\theta_{bs}}{\varkappa_s}\right)\frac{\delta\tau_s}{\delta b} + \frac{\theta_{bs}}{\tau_s\varkappa_s}\left(\bar{\tau}_s^{\;2} + \frac{\varkappa_s^{\;2}}{4}\right)\operatorname{div} \underset{\sim}{b}\,,$$

by (4.1) and (2.41) . (4.5)

From (4.5), on substituting for $\frac{\delta \tau_s}{\delta b}$, the expression obtained by substituting (4.1) into (2.42), one obtains

$$\frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} = - \frac{\theta_{bs}}{\tau_s} \frac{\delta}{\delta b} \left(\varkappa_s + \operatorname{div} \underset{\sim}{n}\right) + \left(\frac{\operatorname{div} \underset{\sim}{b}}{\tau_s} - \frac{2\theta_{bs}}{\varkappa_s}\right) \frac{\delta \theta_{bs}}{\delta n}$$

$$+ \left(\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} + \frac{2\theta_{bs}^{\,2}}{\tau_s \varkappa_s}\right) \frac{\delta \theta_{bs}}{\delta b} + \frac{\theta_{bs}}{\tau_s \varkappa_s} \left(\bar{\tau}_s^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right) \operatorname{div} \underset{\sim}{b}$$

$$- \frac{4\theta_{bs}^{\,2}}{\varkappa_s^{\,2}} \left(\bar{\tau}_s^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right) . \tag{4.6}$$

From (2.47), on substituting for $\frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b}$, $\frac{\delta}{\delta n} (\varkappa_s + \operatorname{div} \underset{\sim}{n})$, and $\varkappa + \operatorname{div} \underset{\sim}{n}$, the expressions (4.6), (4.4) and (4.1), respectively, one obtains

$$\frac{\tau_s}{\theta_{bs}} \frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} + \frac{\theta_{bs}}{\tau_s} \frac{\delta}{\delta b} \left(\varkappa_s + \operatorname{div} \underset{\sim}{n}\right) = \frac{\bar{\tau}_s^{\,2} \operatorname{div} \underset{\sim}{b}}{\tau_s \theta_{bs}^{\,2}} \frac{\delta \theta_{bs}}{\delta n} + \frac{2\bar{\tau}_s^{\,2}}{\tau_s \varkappa_s} \frac{\delta \theta_{bs}}{\delta b}$$

$$- \frac{\bar{\tau}_s^{\,2}}{\theta_{bs}^{\,2}} (\operatorname{div} \underset{\sim}{b})^2$$

$$+ \frac{\bar{\tau}_s^{\,2}}{\tau_s \theta_{bs} \varkappa_s} \left(3\bar{\tau}_s^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right) \operatorname{div} \underset{\sim}{b}$$

$$- \frac{4}{\varkappa_s^{\,2}} \left(\bar{\tau}_s^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right)^2 . \tag{4.7}$$

Taking the $\underset{\sim}{s}$-gradient of (2.43),

$$\frac{\delta^2 \tau_s}{\delta s \delta n} - \frac{\delta^2 \theta_{bs}}{\delta s \delta b} + 2 \frac{\delta \tau_s}{\delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) + 2\tau_s \frac{\delta}{\delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n})$$

$$- 2 \frac{\delta \theta_{bs}}{\delta s} \operatorname{div} \underset{\sim}{b} - 2\theta_{bs} \frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{b} = 0 \ . \tag{4.8}$$

Applying the commutation formulae (2.30) and (2.31) to θ_{bs} and τ_s, respectively,

$$\frac{\delta^2 \theta_{bs}}{\delta s \delta b} = \frac{\delta}{\delta b} \left(\frac{\delta \theta_{bs}}{\delta s}\right) - \theta_{bs} \frac{\delta \theta_{bs}}{\delta b} \ , \tag{4.9}$$

and

$$\frac{\delta^2 \tau_s}{\delta s \delta n} = \frac{\delta}{\delta n} \left(\frac{\delta \tau_s}{\delta s}\right) - \varkappa_s \frac{\delta \tau_s}{\delta s} + \theta_{bs} \frac{\delta \tau_s}{\delta n} + 2\tau_s \frac{\delta \tau_s}{\delta b} \ . \tag{4.10}$$

From (4.8), (4.9), and (4.10), on substituting for $\frac{\delta \tau_s}{\delta s}$, $\frac{\delta}{\delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n})$, $\frac{\delta \theta_{bs}}{\delta s}$, $\frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{b}$, $\frac{\delta \tau_s}{\delta n}$, $\frac{\delta \tau_s}{\delta b}$, and $\varkappa_s + \operatorname{div} \underset{\sim}{n}$, the expressions (2.40), (2.46), (2.39), (2.45), (2.43), (2.42) and (4.1), respectively, one obtains

$$\varkappa_s \left[\frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} - \frac{\delta}{\delta b} (\varkappa_s + \operatorname{div} \underset{\sim}{n})\right] = 4\left(\tau_s \frac{\delta \theta_{bs}}{\delta n} - \theta_{bs} \frac{\delta \theta_{bs}}{\delta b}\right) - 4\tau_s^2 \operatorname{div} \underset{\sim}{b}$$

$$+ \frac{8\tau_s \theta_{bs}}{\varkappa_s} \left(\tau_s^2 - \frac{\varkappa_s^2}{4}\right) , \tag{4.11}$$

a relation, which with (4.7) yields

$$\frac{\delta}{\delta n}\,\mathrm{div}\,\underset{\sim}{b} = \left(\frac{\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}} + \frac{4\tau_s\theta_{bs}^{2}}{\tau_{\bar{s}}^{2}\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta n} - \frac{2\theta_{bs}}{\tau_{\bar{s}}^{2}\varkappa_s}\left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right)\frac{\delta\theta_{bs}}{\delta b}$$

$$- \frac{\tau_s}{\theta_{bs}}\left(\mathrm{div}\,\underset{\sim}{b}\right)^{2} - \frac{1}{\varkappa_s}\left[\tau_{\bar{s}}^{2} - \left(4\tau_s^{2} - \frac{\varkappa_s^{2}}{4}\right)\right]\mathrm{div}\,\underset{\sim}{b}$$

$$+ \frac{4\tau_s\theta_{bs}}{\tau_{\bar{s}}^{2}\varkappa_s^{2}}\left[\tau_{\bar{s}}^{4} - \left(2\tau_s^{2} + \varkappa_s^{2}\right)\tau_{\bar{s}}^{2} + \frac{\varkappa_s^{2}}{2}\left(\tau_s^{2} - \frac{\varkappa_s^{2}}{8}\right)\right], \qquad (4.12)$$

and

$$\frac{\delta}{\delta b}\left(\varkappa_s + \mathrm{div}\,\underset{\sim}{n}\right) = \left(\frac{\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}} - \frac{4\tau_s^{3}}{\tau_{\bar{s}}^{2}\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta n} + \frac{2\theta_{bs}}{\tau_{\bar{s}}^{2}\varkappa_s}\left(\tau_{\bar{s}}^{2} + 2\tau_s^{2}\right)\frac{\delta\theta_{bs}}{\delta b}$$

$$- \frac{\tau_s}{\theta_{bs}}\left(\mathrm{div}\,\underset{\sim}{b}\right)^{2} + \frac{1}{\varkappa_s}\left[3\tau_{\bar{s}}^{2} + \left(4\tau_s^{2} - \frac{\varkappa_s^{2}}{4}\right)\right]\mathrm{div}\,\underset{\sim}{b}$$

$$- \frac{4\tau_s\theta_{bs}}{\tau_{\bar{s}}^{2}\varkappa_s^{2}}\left[4\tau_{\bar{s}}^{4} + 2\tau_s^{2}\tau_{\bar{s}}^{2} + \frac{\varkappa_s^{2}}{2}\left(\theta_{bs}^{2} + \frac{\varkappa_s^{2}}{8}\right)\right]. \qquad (4.13)$$

Substituting (4.12) and (4.13) into (4.4) and (4.6), respectively, one obtains

$$\frac{\delta}{\delta b}\,\mathrm{div}\,\underset{\sim}{b} = -\frac{2\theta_{bs}}{\tau_{\bar{s}}^{2}\varkappa_s}\left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right)\frac{\delta\theta_{bs}}{\delta n} + \left(\frac{\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}} - \frac{4\tau_s\theta_{bs}^{2}}{\tau_{\bar{s}}^{2}\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta b} + \left(\mathrm{div}\,\underset{\sim}{b}\right)^{2}$$

$$- \frac{4\tau_s\theta_{bs}}{\varkappa_s}\,\mathrm{div}\,\underset{\sim}{b} + \frac{4\theta_{bs}^{2}}{\tau_{\bar{s}}^{2}\varkappa_s^{2}}\left[\left(2\tau_s^{2} + \frac{3}{4}\varkappa_s^{2}\right)\tau_{\bar{s}}^{2} - \frac{\varkappa_s^{2}}{2}\left(\tau_s^{2} - \frac{\varkappa_s^{2}}{8}\right)\right] \qquad (4.14)$$

and

$$\frac{\delta}{\delta n}\left(\varkappa_s + \mathrm{div}\,\underset{\sim}{n}\right) = \frac{2\theta_{bs}}{\tau_{\bar{s}}^{2}\varkappa_s}\left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right)\frac{\delta\theta_{bs}}{\delta n} - \left(\frac{\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}} - \frac{4\tau_s\theta_{bs}^{2}}{\tau_{\bar{s}}^{2}\varkappa_s}\right)\frac{\delta\theta_{bs}}{\delta b}$$

$$- \frac{1}{\theta_{bs}^{2}}\left(2\tau_{\bar{s}}^{2} - \tau_s^{2}\right)\left(\mathrm{div}\,\underset{\sim}{b}\right)^{2} + \frac{\tau_s}{\theta_{bs}\varkappa_s}\left[6\tau_{\bar{s}}^{2} - \left(4\tau_s^{2} + \frac{\varkappa_s^{2}}{2}\right)\right]\mathrm{div}\,\underset{\sim}{b}$$

$$- \frac{1}{\tau_{\bar{s}}^{2}\varkappa_s^{2}}\Big(\tau_{\bar{s}}^{6} + \left(8\tau_s^{2} + \tfrac{3}{2}\varkappa_s^{2}\right)\tau_{\bar{s}}^{4}$$

$$- \left(8\tau_s^{4} + 5\tau_s^{2}\varkappa_s^{2} - \tfrac{5}{16}\varkappa_s^{4}\right)\tau_{\bar{s}}^{2} + \tau_s^{2}\varkappa_s^{2}\left(2\tau_s^{2} - \frac{\varkappa_s^{2}}{4}\right)\Big] \quad (4.15)$$

Taking the $\underset{\sim}{s}$-gradient of (2.53),

$$\frac{\delta\tau_{\bar{s}}^{2}}{\delta s} - \left[\frac{\varkappa_s}{2} + \left(\varkappa_s + \mathrm{div}\,\underset{\sim}{n}\right) + \frac{\tau_s\,\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}}\right]\frac{\delta\varkappa_s}{\delta s} - \varkappa_s\frac{\delta}{\delta s}\left(\varkappa_s + \mathrm{div}\,\underset{\sim}{n}\right)$$

$$- \frac{\varkappa\,\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}}\frac{\delta\tau_s}{\delta s} + \frac{\tau_s\varkappa_s\,\mathrm{div}\,\underset{\sim}{b}}{\theta_{bs}^{2}}\frac{\delta\theta_{bs}}{\delta s} - \frac{\tau_s\varkappa_s}{\theta_{bs}}\frac{\delta}{\delta s}\,\mathrm{div}\,\underset{\sim}{b} = 0\,. \quad (4.16)$$

From (4.16), on substituting for $\frac{\delta\tau_{\bar{s}}^{2}}{\delta s}$, $\frac{\delta\varkappa_s}{\delta s}$, $\frac{\delta}{\delta s}\left(\varkappa_s + \mathrm{div}\,\underset{\sim}{n}\right)$, $\frac{\delta\tau_s}{\delta s}$, $\frac{\delta\theta_{bs}}{\delta s}$, and $\frac{\delta}{\delta s}\,\mathrm{div}\,\underset{\sim}{b}$, the expressions (2.55), (2.33), (2.48), (2.40), (2.39), and (2.45), respectively, and substituting for $\varkappa_s + \mathrm{div}\,\underset{\sim}{n}$ the expression (4.1), one obtains

$$\varkappa_s\left[\frac{\delta\theta_{bs}}{\delta n} + \frac{\tau_s}{\theta_{bs}}\frac{\delta\theta_{bs}}{\delta b} - \frac{\tau_{\bar{s}}^{2}\varkappa_s}{\theta_{bs}^{3}}(\mathrm{div}\ \underset{\sim}{b})^2 + \frac{\tau_s}{\theta_{bs}^{2}}\left(2\tau_{\bar{s}}^{2} - \frac{\varkappa_s^2}{4}\right)\mathrm{div}\ \underset{\sim}{b}\right.$$

$$\left. + \frac{2\theta_{bs}}{\varkappa_s}\left(\tau_{\bar{s}}^{2} - \frac{\varkappa_s^2}{4}\right)\right] = 0 \qquad (4.17)$$

Since θ_{bs} and $\varkappa_s$ are different from zero, (4.17) reduces to the following form:

$$\theta_{bs}\ \frac{\delta\theta_{bs}}{\delta n} + \tau_s\ \frac{\delta\theta_{bs}}{\delta b} - \frac{\tau_{\bar{s}}^{2}\varkappa_s}{\theta_{bs}^{2}}(\mathrm{div}\ \underset{\sim}{b})^2$$

$$+ \frac{\tau_s}{\theta_{bs}}\left(2\tau_{\bar{s}}^{2} - \frac{\varkappa_s^2}{4}\right)\mathrm{div}\ \underset{\sim}{b} + \frac{2\theta_{bs}^{2}}{\varkappa_s}\left(\tau_{\bar{s}}^{2} - \frac{\varkappa_s^2}{4}\right) = 0 \quad . \qquad (4.18)$$

Taking the $\underset{\sim}{s}$-gradient of (4.18),

$$\theta_{bs}\frac{\delta^2\theta_{bs}}{\delta s\delta n} + \tau_s\ \frac{\delta^2\theta_{bs}}{\delta s\delta b} + \left[\frac{\delta\theta_{bs}}{\delta n} + \frac{2\tau_{\bar{s}}^{2}\varkappa_s}{\theta_{bs}^{3}}(\mathrm{div}\ \underset{\sim}{b})^2 - \frac{\tau_s}{\theta_{bs}^{2}}\left(2\tau_{\bar{s}}^{2} - \frac{\varkappa_s^2}{4}\right)\mathrm{div}\ \underset{\sim}{b}\right.$$

$$\left. + \frac{4\theta_{bs}}{\varkappa_s}\left(\tau_{\bar{s}}^{2} - \frac{\varkappa_s^2}{4}\right)\right]\frac{\delta\theta_{bs}}{\delta s} + \left[\frac{\delta\theta_{bs}}{\delta b} + \frac{\mathrm{div}\ \underset{\sim}{b}}{\theta_{bs}}\left(2\tau_{\bar{s}}^{2} - \frac{\varkappa_s^2}{4}\right)\right]\frac{\delta\tau_s}{\delta s}$$

$$- \left[\frac{\tau_{\bar{s}}^{2}}{\theta_{bs}^{2}}(\mathrm{div}\ \underset{\sim}{b})^2 + \frac{\tau_s\varkappa_s}{2\theta_{bs}}\mathrm{div}\ \underset{\sim}{b} + \frac{2\theta_{bs}^{2}}{\varkappa_s^{2}}\left(\tau_{\bar{s}}^{2} + \frac{\varkappa_s^2}{4}\right)\right]\frac{\delta\varkappa_s}{\delta s}$$

$$- \left[\frac{\varkappa_s}{\theta_{bs}^{2}}(\mathrm{div}\ \underset{\sim}{b})^2 - \frac{2\tau_s}{\theta_{bs}}\mathrm{div}\ \underset{\sim}{b} - \frac{2\theta_{bs}^{2}}{\varkappa_s}\right]\frac{\delta\tau_{\bar{s}}^{2}}{\delta s}$$

$$- \left[\frac{2\tau_{\bar{s}}^{2}\varkappa_s}{\theta_{bs}^{2}}\mathrm{div}\ \underset{\sim}{b} - \frac{\tau_s}{\theta_{bs}}\left(2\tau_{\bar{s}}^{2} - \frac{\varkappa_s^2}{4}\right)\right]\frac{\delta}{\delta s}\mathrm{div}\ \underset{\sim}{b} = 0 \quad . \qquad (4.19)$$

Applying the commutation formulae (2.30) and (2.31) to θ_{bs}, one has

$$\frac{\delta^2\theta_{bs}}{\delta s\delta b} = \frac{\delta^2\theta_{bs}}{\delta b\delta s} - \theta_{bs}\frac{\delta\theta_{bs}}{\delta b}\,, \tag{4.20}$$

and

$$\frac{\delta^2\theta_{bs}}{\delta s\delta n} = \frac{\delta^2\theta_{bs}}{\delta n\delta s} - \varkappa_s\frac{\delta\theta_{bs}}{\delta s} + \theta_{bs}\frac{\delta\theta_{bs}}{\delta n} + 2\tau_s\frac{\delta\theta_{bs}}{\delta b}\,. \tag{4.21}$$

On substituting for $\dfrac{\delta^2\theta_{bs}}{\delta b\delta s}$ and $\dfrac{\delta^2\theta_{bs}}{\delta n\delta s}$ the expressions obtained by taking the $\underset{\sim}{b}$ and $\underset{\sim}{n}$ gradients of (2.39), and then substituting for $\dfrac{\delta\varkappa_s}{\delta b}$, $\dfrac{\delta\tau_s}{\delta b}$, $\dfrac{\delta\tau_s}{\delta n}$, and $\dfrac{\delta\varkappa_s}{\delta n}$, the expressions (2.41), (2.42), (2.43), and (2.44), respectively, (4.20) and (4.21) become

$$\frac{\delta^2\theta_{bs}}{\delta s\delta b} = \left[\frac{\varkappa_s}{\theta_{bs}}\operatorname{div}\underset{\sim}{b} - \frac{2\tau_s}{\tau_{\bar{s}}^{\,2}}\left(\tau_{\bar{s}}^{\,2} + 2\tau_s^{\,2}\right)\right]\frac{\delta\theta_{bs}}{\delta n} - \frac{\theta_{bs}}{\tau_{\bar{s}}^{\,2}}\left(\tau_{\bar{s}}^{\,2} - 4\tau_s^{\,2}\right)\frac{\delta\theta_{bs}}{\delta b}$$

$$+ \left(2\tau_{\bar{s}}^{\,2} + 4\tau_s^{\,2}\right)\operatorname{div}\underset{\sim}{b}$$

$$- \frac{4\tau_s\theta_{bs}}{\tau_{\bar{s}}^{\,2}\varkappa_s}\left[2\tau_{\bar{s}}^{\,4} + \left(2\tau_s^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right)\tau_{\bar{s}}^{\,2} - \frac{\varkappa_s^{\,2}}{2}\left(\tau_s^{\,2} - \frac{\varkappa_s^{\,2}}{8}\right)\right], \tag{4.22}$$

$$\frac{\delta^2\theta_{bs}}{\delta s\delta n} = \frac{\theta_{bs}}{\tau_{\bar{s}}^{\,2}}\left(\tau_{\bar{s}}^{\,2} - 4\tau_s^{\,2}\right)\frac{\delta\theta_{bs}}{\delta n} - \left[\frac{\varkappa_s}{\theta_{bs}}\operatorname{div}\underset{\sim}{b} - \frac{4\tau_s}{\tau_{\bar{s}}^{\,2}}\left(2\tau_{\bar{s}}^{\,2} - \tau_s^{\,2}\right)\right]\frac{\delta\theta_{bs}}{\delta b}$$

$$- \frac{2\tau_{\bar{s}}^{\,2}\varkappa_s}{\theta_{bs}^{\,2}}\left(\operatorname{div}\underset{\sim}{b}\right)^2 + \frac{\tau_s}{\theta_{bs}}\left(6\tau_{\bar{s}}^{\,2} + 4\theta_{bs}^{\,2} - \varkappa_s^{\,2}\right)\operatorname{div}\underset{\sim}{b}$$

$$- \frac{1}{\tau_{\bar{s}}^{2}\varkappa_{s}}\left[\left(4\tau_{s}^{2} + \frac{\varkappa_{s}^{2}}{2}\right)\tau_{\bar{s}}^{4} + \left(8\tau_{s}^{2}\theta_{bs}^{2} - 2\tau_{s}^{2}\varkappa_{s}^{2} + \frac{3}{8}\varkappa_{s}^{4}\right)\tau_{\bar{s}}^{2}\right.$$

$$\left. - \tau_{s}^{2}\varkappa_{s}^{2}\left(2\theta_{bs}^{2} + \frac{\varkappa_{s}^{2}}{4}\right)\right] . \tag{4.23}$$

From (4.19), on substituting for $\frac{\delta^2\theta_{bs}}{\delta s\delta n}$, $\frac{\delta^2\theta_{bs}}{\delta s\delta b}$, $\frac{\delta\theta_{bs}}{\delta s}$, $\frac{\delta\tau_s}{\delta s}$, $\frac{\delta\varkappa_s}{\delta s}$, $\frac{\delta\tau_{\bar{s}}^2}{\delta s}$, and $\frac{\delta}{\delta s}\operatorname{div}\underset{\sim}{b}$, the expressions (4.23), (4.22), (2.39), (2.40), (2.33), (2.55), and (2.45), respectively, and reducing by (4.18), one obtains

$$\left(4\tau_{s}^{2} + \frac{\varkappa_{s}^{2}}{4}\right)\frac{\delta\theta_{bs}}{\delta n} - \left[\frac{2\tau_{\bar{s}}^{2}\varkappa_{s}}{\theta_{bs}^{2}}\operatorname{div}\underset{\sim}{b} + \frac{\tau_{s}}{\theta_{bs}}\left(2\tau_{\bar{s}}^{2} - 4\tau_{s}^{2} + \frac{\varkappa_{s}^{2}}{4}\right)\right]\frac{\delta\theta_{bs}}{\delta b}$$

$$+ \frac{2\tau_{\bar{s}}^{2}\tau_{s}\varkappa_{s}^{2}}{\theta_{bs}^{4}}(\operatorname{div}\underset{\sim}{b})^{3} - \frac{\varkappa_{s}}{\theta_{bs}^{3}}\left(2\tau_{\bar{s}}^{2} - \frac{\varkappa_{s}^{2}}{4}\right)(\tau_{\bar{s}}^{2} + 2\tau_{s}^{2})(\operatorname{div}\underset{\sim}{b})^{2}$$

$$+ \frac{\tau_{s}}{\theta_{bs}^{2}}\left[4\tau_{s}^{2}\tau_{\bar{s}}^{2} - \varkappa_{s}^{2}\left(\tau_{s}^{2} - \frac{\varkappa_{s}^{2}}{16}\right)\right]\operatorname{div}\underset{\sim}{b}$$

$$+ \frac{\theta_{bs}}{\varkappa_{s}}\left[4\tau_{\bar{s}}^{4} + \left(8\tau_{s}^{2} + \frac{5}{2}\varkappa_{s}^{2}\right)\tau_{\bar{s}}^{2} - \varkappa_{s}^{2}\left(2\tau_{s}^{2} - \frac{\varkappa_{s}^{2}}{8}\right)\right] = 0 \; . \tag{4.24}$$

A check of the condition (4.24) is given in the next chapter. The condition (5.21) of the next chapter is verified by alternative computational procedures. This in turn gives a check on (4.24).

The relation (4.18) and (4.24) are now solved for $\frac{\delta\theta_{bs}}{\delta n}$ and $\frac{\delta\theta_{bs}}{\delta b}$. One obtains

$$\frac{\delta\theta_{bs}}{\delta n} = \frac{1}{D}\Big[\frac{\tau_{\bar{s}}^{2}\kappa_s^{2}}{\theta_{bs}^{2}}(\operatorname{div}\mathbf{b})^3 + \frac{\tau_s\kappa_s^{3}}{4\theta_{bs}}(\operatorname{div}\mathbf{b})^2 - 2\tau_{\bar{s}}^{2}\Big(\tau_{\bar{s}}^{2} + \tau_s^{2} - \frac{\kappa_s^{2}}{4}\Big)\operatorname{div}\mathbf{b}$$

$$- \frac{4\tau_{\bar{s}}^{2}\tau_s\theta_{bs}}{\kappa_s}\Big(\tau_{\bar{s}}^{2} + \frac{\kappa_s^{2}}{4}\Big)\Big], \qquad (4.25)$$

and

$$\frac{\delta\theta_{bs}}{\delta b} = \frac{1}{D}\Big\{\frac{\tau_{\bar{s}}^{2}\tau_s\kappa_s^{2}}{\theta_{bs}^{3}}(\operatorname{div}\mathbf{b})^3 - \frac{\kappa_s}{\theta_{bs}^{2}}\Big(\tau_{\bar{s}}^{4} - \frac{\kappa_s^{2}}{4}\tau_{\bar{s}}^{2} - \frac{\tau_s^{2}\kappa_s^{2}}{4}\Big)(\operatorname{div}\mathbf{b})^2$$

$$- \frac{\tau_s}{\theta_{bs}}\Big[\Big(2\tau_s^{2} + \frac{\kappa_s^{2}}{4}\Big)\tau_{\bar{s}}^{2} - \frac{\kappa_s^{4}}{16}\Big]\operatorname{div}\mathbf{b} + \frac{2\theta_{bs}^{2}}{\kappa_s}\Big(\tau_{\bar{s}}^{2} + \frac{\kappa_s^{2}}{4}\Big)^{2}\Big\}. \qquad (4.26)$$

It follows from Lemma 3.5 that D must be non-vanishing.

The condition (4.19) is verified in the Appendix. The conditions (4.25) and (4.26) are checked as follows. One may compute the $\mathbf{s}$-gradients of (4.25) and (4.26) directly, to obtain explicit expressions for $\frac{\delta^2\theta_{bs}}{\delta s\delta n}$ and $\frac{\delta^2\theta_{bs}}{\delta s\delta b}$. A substitution of these expressions in (4.19) shows the latter condition to be identically satisfied.

When (4.25) and (4.26) are substituted in (4.18), one obtains an identity. When they are substituted back in (4.4) and (4.6) one obtains relations between $\frac{\delta}{\delta n}(\kappa_s + \operatorname{div}\mathbf{n})$ and $\frac{\delta}{\delta n}\operatorname{div}\mathbf{b}$, and between $\frac{\delta}{\delta b}(\kappa_s + \operatorname{div}\mathbf{n})$ and $\frac{\delta}{\delta b}\operatorname{div}\mathbf{b}$. The same relations are also obtained from conditions (4.12), (4.13), (4.14), and (4.15). Inasmuch as the conditions (4.18), (4.4), and (4.6) are obtained by taking the $\mathbf{s}$, $\mathbf{n}$, and $\mathbf{b}$ gradients of Hamel's condition (2.53), one sees that no new relations can be obtained by taking higher gradients of the relation (2.53).

By (4.1), one may eliminate the parameter $\varkappa_s + \operatorname{div} \underset{\sim}{n}$, and only deal with four parameters θ_{bs}, τ_s, $\varkappa_s$, and $\operatorname{div} \underset{\sim}{b}$. The gradients of these four parameters can be tabulated as follows:

$$\frac{\delta\theta_{bs}}{\delta s} = -\frac{\tau_s\varkappa_s}{\theta_{bs}} \operatorname{div} \underset{\sim}{b} + 2\tau_s^{\,2} - \frac{\varkappa_s^{\,2}}{4} \tag{4.27}$$

$$\begin{aligned}\frac{\delta\theta_{bs}}{\delta n} = \frac{1}{D}\Bigg[&\frac{\tau_{\bar{s}}^{\,2}\varkappa_s^{\,2}}{\theta_{bs}^{\,2}} (\operatorname{div} \underset{\sim}{b})^3 + \frac{\tau_s\varkappa_s^{\,3}}{4\theta_{bs}} (\operatorname{div} \underset{\sim}{b})^2 - 2\tau_{\bar{s}}^{\,2}\left(\tau_{\bar{s}}^{\,2} + \tau_s^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right)\operatorname{div} \underset{\sim}{b} \\ &- \frac{4\tau_{\bar{s}}^{\,2}\tau_s\theta_{bs}}{\varkappa_s}\left(\tau_{\bar{s}}^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right)\Bigg]\end{aligned} \tag{4.28}$$

$$\begin{aligned}\frac{\delta\theta_{bs}}{\delta b} = \frac{1}{D}\Bigg\{&\frac{\tau_{\bar{s}}^{\,2}\tau_s\varkappa_s^{\,2}}{\theta_{bs}^{\,3}} (\operatorname{div} \underset{\sim}{b})^3 - \frac{\varkappa_s}{\theta_{bs}^{\,2}}\left(\tau_{\bar{s}}^{\,4} - \frac{\varkappa_s^{\,2}}{4}\tau_{\bar{s}}^{\,2} - \frac{\tau_s^{\,2}\varkappa_s^{\,2}}{4}\right)(\operatorname{div} \underset{\sim}{b})^2 \\ &- \frac{\tau_s}{\theta_{bs}}\left[\left(2\tau_s^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right)\tau_{\bar{s}}^{\,2} - \frac{\varkappa_s^{\,4}}{16}\right]\operatorname{div} \underset{\sim}{b} + \frac{2\theta_{bs}^{\,2}}{\varkappa_s}\left(\tau_{\bar{s}}^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right)^2\Bigg\}\end{aligned} \tag{4.29}$$

$$\frac{\delta\tau_s}{\delta s} = \varkappa_s \operatorname{div} \underset{\sim}{b} - 2\tau_s\theta_{bs} \tag{4.30}$$

$$\frac{\delta\tau_s}{\delta n} = \frac{\delta\theta_{bs}}{\delta b} + \frac{2\tau_{\bar{s}}^{\,2}}{\theta_{bs}} \operatorname{div} \underset{\sim}{b} - \frac{2\tau_s}{\varkappa_s}\left(\tau_{\bar{s}}^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right) \tag{4.31}$$

$$\frac{\delta\tau_s}{\delta b} = -\frac{\delta\theta_{bs}}{\delta n} - \frac{2\theta_{bs}}{\varkappa_s}\left(\tau_{\bar{s}}^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right) \tag{4.32}$$

$$\frac{\delta \varkappa_s}{\delta s} = \theta_{bs} \varkappa_s \tag{4.33}$$

$$\frac{\delta \varkappa_s}{\delta n} = \frac{\tau_s \varkappa_s}{\theta_{bs}} \operatorname{div} \underset{\sim}{b} + \tau_{\bar{s}}^2 + \frac{5}{4} \varkappa_s^2 \tag{4.34}$$

$$\frac{\delta \varkappa_s}{\delta b} = - \varkappa_s \operatorname{div} \underset{\sim}{b} \tag{4.35}$$

$$\frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{b} = - \frac{\delta \theta_{bs}}{\delta b} - \theta_{bs} \operatorname{div} \underset{\sim}{b} \tag{4.36}$$

$$\begin{aligned} \frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} = {} & \left(\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} + \frac{4\tau_s \theta_{bs}^2}{\tau_{\bar{s}}^2 \varkappa_s} \right) \frac{\delta \theta_{bs}}{\delta n} - \frac{2\theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s} \left(\tau_{\bar{s}}^2 - 2\tau_s^2 \right) \frac{\delta \theta_{bs}}{\delta b} \\ & - \frac{\tau_s}{\theta_{bs}} (\operatorname{div} \underset{\sim}{b})^2 - \frac{1}{\varkappa_s} \left[\tau_{\bar{s}}^2 - \left(4\tau_s^2 - \frac{\varkappa_s^2}{4} \right) \right] \operatorname{div} \underset{\sim}{b} \\ & + \frac{4\tau_s \theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s^2} \left[\tau_{\bar{s}}^4 - (2\tau_s^2 + \varkappa_s^2) \tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{2} \left(\tau_s^2 - \frac{\varkappa_s^2}{8} \right) \right] \end{aligned} \tag{4.37}$$

$$\begin{aligned} \frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} = {} & - \frac{2\theta_{bs}}{\tau_{\bar{s}}^2 \varkappa_s} \left(\tau_{\bar{s}}^2 - 2\tau_s^2 \right) \frac{\delta \theta_{bs}}{\delta n} + \left(\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} - \frac{4\tau_s \theta_{bs}^2}{\tau_{\bar{s}}^2 \varkappa_s} \right) \frac{\delta \theta_{bs}}{\delta b} + (\operatorname{div} \underset{\sim}{b})^2 \\ & - \frac{4\tau_s \theta_{bs}}{\varkappa_s} \operatorname{div} \underset{\sim}{b} + \frac{4\theta_{bs}^2}{\tau_{\bar{s}}^2 \varkappa_s^2} \left[\left(2\tau_s^2 + \frac{3}{4}\varkappa_s^2 \right) \tau_{\bar{s}}^2 \right. \\ & \left. - \frac{\varkappa_s^2}{2} \left(\tau_s^2 - \frac{\varkappa_s^2}{8} \right) \right] . \end{aligned} \tag{4.38}$$

One can conclude from the above expressions that since $\frac{\delta\theta_{bs}}{\delta n}$ and $\frac{\delta\theta_{bs}}{\delta b}$ are expressed as polynomials in θ_{bs}, τ_s, κ_s, and div $\underset{\sim}{b}$, all the gradients of the parameters θ_{bs}, τ_s, κ_s, and div $\underset{\sim}{b}$ can be expressed as polynomials in these variables. It is thus merely a matter of algebra to manipulate the compatibility conditions of Chapter 3 to isolate the polynomials needed to establish the contradictions necessary for proving Hamel's theorem.

CHAPTER V

THREE POLYNOMIALS

The parameters θ_{bs}, τ_s, $\varkappa_s$, and div $\underset{\sim}{b}$ must satisfy the identity (2.28). In other words, one may obtain new relations by applying the commutation formulae (2.29), (2.30), and (2.31) to each of θ_{bs}, τ_s, $\varkappa_s$, and div $\underset{\sim}{b}$. We obtain explicitly twelve relations. However one can only obtain three polynomial relations by substituting the expressions (4.27) $\sim$ (4.38) to the corresponding terms of these twelve relations. This is proved by investigating the twelve relations one by one in the Appendix.

In order to derive the first polynomial one applies the commutation formula (2.29) to θ_{bs},

$$\frac{\delta^2\theta_{bs}}{\delta b\,\delta n} - \frac{\delta^2\theta_{bs}}{\delta n\,\delta b} = -\,\mathrm{div}\,\underset{\sim}{b}\,\frac{\delta\theta_{bs}}{\delta n} + (\varkappa_s + \mathrm{div}\,\underset{\sim}{n})\,\frac{\delta\theta_{bs}}{\delta b}\ . \tag{5.1}$$

From (5.1), on substituting for $\frac{\delta^2\theta_{bs}}{\delta b\delta n}$ and $\frac{\delta^2\theta_{bs}}{\delta n\,\delta b}$, the expressions obtained by taking the $\underset{\sim}{b}$ and $\underset{\sim}{n}$ gradients of (4.28) and (4.29), respectively, and then substituting for $\frac{\delta\theta_{bs}}{\delta n}$, $\frac{\delta\theta_{bs}}{\delta b}$, $\frac{\delta\tau_s}{\delta n}$, $\frac{\delta\tau_s}{\delta b}$, $\frac{\delta\varkappa_s}{\delta n}$, $\frac{\delta\varkappa_s}{\delta b}$, $\frac{\delta}{\delta n}$ div $\underset{\sim}{b}$, and $\frac{\delta}{\delta b}$ div $\underset{\sim}{b}$, the expressions (4.28), (4.29), (4.31), (4.32), (4.34), (4.35), (4.37), and (4.38), respectively, one obtains

$$\frac{1}{\varkappa_s^{\,2}D^3}\sum_{i=0}^{6} A_i\,x^i = 0\ , \tag{5.2}$$

where

$$x = \frac{\varkappa_s}{\theta_{bs}} \operatorname{div} b \;, \tag{5.3}$$

$$\begin{aligned} A_0 = \frac{2}{\tau_{\bar s}^2}\left(\tau_{\bar s}^2 + \frac{\varkappa_s^2}{4}\right)^3 \Big[& 2\tau_{\bar s}^{10} + 2\tau_s^2\tau_{\bar s}^8 + \frac{\varkappa_s^2}{2}\left(5\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\tau_{\bar s}^6 \\ & - \tau_s^2\varkappa_s^2\left(\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\tau_{\bar s}^4 - \frac{\tau_s^4\varkappa_s^4}{8}\,\tau_{\bar s}^2 - \frac{1}{32}\,\tau_s^4\varkappa_s^6\Big] \;, \end{aligned} \tag{5.4}$$

$$\begin{aligned} A_1 = \frac{\tau_s}{\tau_{\bar s}^2}\left(\tau_{\bar s}^2 + \frac{\varkappa_s^2}{4}\right)\Big[& 8\tau_{\bar s}^{12} + \left(8\tau_s^2 + \frac{9}{2}\varkappa_s^2\right)\tau_{\bar s}^{10} + \varkappa_s^2\left(2\tau_s^2 + \frac{9}{8}\varkappa_s^2\right)\tau_{\bar s}^8 \\ & - \frac{3}{4}\varkappa_s^4\left(\tau_s^2 - \frac{\varkappa_s^2}{8}\right)\tau_{\bar s}^6 - \frac{\varkappa_s^4}{4}\left(3\tau_s^4 + \tau_s^2\varkappa_s^2 + \frac{\varkappa_s^4}{32}\right)\tau_{\bar s}^4 \\ & - \frac{\tau_s^2\varkappa_s^6}{4}\left(\tau_s^2 + \frac{3}{16}\varkappa_s^2\right)\tau_{\bar s}^2 + \frac{\tau_s^4\varkappa_s^8}{64}\Big] \;, \end{aligned} \tag{5.5}$$

$$\begin{aligned} A_2 = & -4\tau_{\bar s}^{12} + \left(6\tau_s^2 + \frac{\varkappa_s^2}{2}\right)\tau_{\bar s}^{10} - \varkappa_s^2\left(6\tau_s^2 - \frac{7}{8}\varkappa_s^2\right)\tau_{\bar s}^8 \\ & - \varkappa_s^2\left(\tau_s^4 + \frac{15}{8}\tau_s^2\varkappa_s^2 - \frac{3}{32}\varkappa_s^4\right)\tau_{\bar s}^6 - \frac{\varkappa_s^4}{4}\left(11\tau_s^4 + \frac{3}{2}\tau_s^2\varkappa_s^2 + \frac{\varkappa_s^4}{32}\right)\tau_{\bar s}^4 \\ & - \frac{\tau_s^2\varkappa_s^6}{8}\left(\tau_s^2 + \frac{3}{4}\varkappa_s^2\right)\tau_{\bar s} - \frac{\tau_s^4\varkappa_s^6}{8}\left(\tau_s^2 - \frac{3}{4}\varkappa_s^2\right) \;, \end{aligned} \tag{5.6}$$

$$\begin{aligned} A_3 = -\tau_s\Big[& 8\tau_{\bar s}^{10} + \left(4\tau_s^2 + \frac{11}{4}\varkappa_s^2\right)\tau_{\bar s}^8 + \frac{\varkappa_s^2}{2}\left(11\tau_s^2 + \frac{5}{4}\varkappa_s^2\right)\tau_{\bar s}^6 \\ & - \varkappa_s^4\left(\tau_s^2 - \frac{23}{64}\varkappa_s^2\right)\tau_{\bar s}^4 + \frac{\tau_s^2\varkappa_s^4}{4}\left(3\tau_s^2 - \frac{21}{8}\varkappa_s^2\right)\tau_{\bar s}^2 + \frac{1}{16}\tau_s^4\varkappa_s^6\Big], \end{aligned} \tag{5.7}$$

$$A_4 = -\tau_{\bar{s}}^{2}\Big[\tau_{\bar{s}}^{8} + 4\tau_s^{2}\tau_{\bar{s}}^{6} - \frac{\varkappa_s^{2}}{2}\left(5\tau_s^{2} - \tfrac{7}{8}\varkappa_s^{2}\right)\tau_{\bar{s}}^{4} + \tau_s^{2}\varkappa_s^{2}\left(\tau_s^{2} - \tfrac{15}{8}\varkappa_s^{2}\right)\tau_{\bar{s}}^{2}$$

$$+ \tfrac{3}{8}\,\tau_s^{4}\varkappa_s^{4}\Big], \tag{5.8}$$

$$A_5 = \tau_{\bar{s}}^{4}\tau_s\left(2\tau_{\bar{s}}^{4} + \tfrac{9}{4}\varkappa_s^{2}\tau_{\bar{s}}^{2} - \frac{\tau_s^{2}\varkappa_s^{2}}{2}\right), \tag{5.9}$$

$$A_6 = \tau_{\bar{s}}^{8}\ . \tag{5.10}$$

Since $\varkappa_s$ and D do not vanish by virtue of Lemma 3.1 and Lemma 3.5, the first polynomial is obtained as follows:

$$\sum_{i=0}^{6} A_i\, x^i = 0\ . \tag{5.11}$$

It is the occurrence of D in the denominator of the expressions for $\frac{\delta\theta_{bs}}{\delta n}$ and $\frac{\delta\theta_{bs}}{\delta b}$, that makes the final polynomials so complicated.

In order to verify the polynomial (5.11), it is derived by an alternative procedure as follows.

With non-vanishing of θ_{bs} and τ_s one obtains from (4.18)

$$\frac{\delta^2\theta_{bs}}{\delta b\delta n} = -\frac{\tau_s}{\theta_{bs}}\frac{\delta^2\theta_{bs}}{\delta b^2} + \left(\frac{\tau_s}{\theta_{bs}^{2}}\frac{\delta\theta_{bs}}{\delta b} - \frac{1}{\theta_{bs}}\frac{\delta\tau_s}{\delta b}\right)\frac{\delta\theta_{bs}}{\delta b}$$

$$-\left[\frac{2\tau_{\bar{s}}^{2}\varkappa_s}{\theta_{bs}^{3}} - \frac{\tau_s}{\theta_{bs}^{2}}\left(2\tau_{\bar{s}}^{2} - \frac{\varkappa_s^{2}}{4}\right)\right]\frac{\delta}{\delta b}\operatorname{div}\underset{\sim}{b}$$

$$-\left[\frac{\varkappa_s}{\theta_{bs}^{4}}\left(\tau_{\bar{s}}^{2} + 2\tau_s^{2}\right)(\operatorname{div}\underset{\sim}{b})^{2} - \frac{\tau_s}{\theta_{bs}^{3}}\left(2\tau_s^{2} - \frac{\varkappa_s^{2}}{4}\right)\operatorname{div}\underset{\sim}{b}\right.$$

$$+ \frac{2}{\kappa_s}\left(\bar{\tau}_s^2 + 2\theta_{bs}^2 - \frac{\kappa_s^2}{4}\right)\Big]\frac{\delta\theta_{bs}}{\delta b} + \Big[\frac{2\tau_s\kappa_s}{\theta_{bs}^3}(\mathrm{div}\ \underset{\sim}{b})^2$$

$$- \frac{1}{\theta_{bs}^2}\left(2\bar{\tau}_s^2 + 4\tau_s^2 - \frac{\kappa_s^2}{4}\right)\mathrm{div}\ \underset{\sim}{b} - \frac{4\tau_s\theta_{bs}}{\kappa_s}\Big]\frac{\delta\tau_s}{\delta b}$$

$$+ \Big[\frac{\bar{\tau}_s^2}{\theta_{bs}^3}(\mathrm{div}\ \underset{\sim}{b})^2 + \frac{\tau_s\kappa_s}{2\theta_{bs}^2}\mathrm{div}\ \underset{\sim}{b} - \frac{2\theta_{bs}}{\kappa_s^2}\left(\bar{\tau}_s^2 + \frac{\kappa_s^2}{4}\right)\Big]\frac{\delta\kappa_s}{\delta b}\ , \qquad (5.12)$$

and

$$\frac{\delta^2\theta_{bs}}{\delta n\delta b} = - \frac{\theta_{bs}}{\tau_s}\frac{\delta^2\theta_{bs}}{\delta n^2} + \left(\frac{\theta_{bs}}{\tau_s^2}\frac{\delta\tau_s}{\delta n} - \frac{1}{\tau_n}\frac{\delta\theta_{bs}}{\delta n}\right)\frac{\delta\theta_{bs}}{\delta n}$$

$$- \Big[\frac{2\bar{\tau}_s^2\kappa_s}{\tau_s\theta_{bs}^2}\mathrm{div}\ \underset{\sim}{b} - \frac{1}{\theta_{bs}}\left(2\bar{\tau}_s^2 - \frac{\kappa_s^2}{4}\right)\Big]\frac{\delta}{\delta n}\mathrm{div}\ \underset{\sim}{b}$$

$$+ \Big[\frac{2\tau_s\kappa_s}{\theta_{bs}^3}(\mathrm{div}\ \underset{\sim}{b})^2 + \frac{1}{\theta_{bs}^2}\left(2\bar{\tau}_s^2 - 4\theta_{bs}^2 - \frac{\kappa_s^2}{4}\right)\mathrm{div}\ \underset{\sim}{b}$$

$$- \frac{4\theta_{bs}}{\tau_s\kappa_s}\left(\bar{\tau}_s^2 + \theta_{bs}^2 - \frac{\kappa_s^2}{4}\right)\Big]\frac{\delta\theta_{bs}}{\delta n}$$

$$- \Big[\frac{\bar{\tau}_s^2\kappa_s}{\tau_s^2\theta_{bs}^2}(\mathrm{div}\ \underset{\sim}{b})^2 + \frac{4\tau_s}{\theta_{bs}}\mathrm{div}\ \underset{\sim}{b} - \frac{2\theta_{bs}^2}{\tau_s^2\kappa_s}\left(\bar{\tau}_s^2 - 2\tau_s^2 - \frac{\kappa_s^2}{4}\right)\Big]\frac{\delta\tau_s}{\delta n}$$

$$+ \Big[\frac{\bar{\tau}_s^2}{\tau_s\theta_{bs}^2}(\mathrm{div}\ \underset{\sim}{b})^2 + \frac{\kappa_s}{2\theta_{bs}}\mathrm{div}\ \underset{\sim}{b} + \frac{2\theta_{bs}^2}{\tau_s\kappa_s^2}\left(\bar{\tau}_s^2 + \frac{\kappa_s^2}{4}\right)\Big]\frac{\delta\kappa_s}{\delta n}\ . \qquad (5.13)$$

Subtracting (5.13) from (5.12), and substituting for $\frac{\delta^2 \theta_{bs}}{\delta n^2}$, and $\frac{\delta^2 \theta_{bs}}{\delta b^2}$, the expressions obtained by taking the $\underset{\sim}{n}$ and $\underset{\sim}{b}$ gradients of (4.28) and (4.29), respectively, and then substituting for $\frac{\delta \tau_s}{\delta n}$, $\frac{\delta \tau_s}{\delta b}$, $\frac{\delta \varkappa_s}{\delta n}$, $\frac{\delta \varkappa_s}{\delta b}$, $\frac{\delta}{\delta n}$ div $\underset{\sim}{b}$, $\frac{\delta}{\delta b}$ div $\underset{\sim}{b}$, and $\frac{\delta^2 \theta_{bs}}{\delta b \delta n} - \frac{\delta^2 \theta_{bs}}{\delta n \delta b}$, the expressions (4.31), (4.32), (4.34), (4.35), (4.37), (4.38), and (5.1), respectively, and finally substituting for $\frac{\delta \theta_{bs}}{\delta n}$ and $\frac{\delta \theta_{bs}}{\delta b}$, the expressions (4.28) and (4.29), respectively, one obtains again the polynomial (5.2). This not only verifies the polynomial (5.11), but also supports the idea that no new relations can be obtained by taking higher gradients of Hamel's condition (2.53).

From (4.22), which is a relation obtained by applying the commutation formula (2.30) to θ_{bs}, on substituting for $\frac{\delta^2 \theta_{bs}}{\delta s \delta b}$ the expression obtained by taking the s-gradient of (4.29), and substituting for $\frac{\delta \theta_{bs}}{\delta s}$, $\frac{\delta \tau_s}{\delta s}$, $\frac{\delta \varkappa_s}{\delta s}$, and $\frac{\delta}{\delta s}$ div $\underset{\sim}{b}$ the expressions (4.27), (4.30), (4.33), and (4.36), respectively, and then substituting for $\frac{\delta \theta_{bs}}{\delta n}$ and $\frac{\delta \theta_{bs}}{\delta b}$, the expressions (4.28) and (4.29), respectively, one obtains

$$\frac{\theta_{bs}}{\varkappa_s D^3} \sum_{i=0}^{5} B_i \, x^i = 0 , \tag{5.14}$$

where

$$B_0 = \frac{\left(\tau_s^{-2} + \frac{\varkappa_s^2}{4}\right)^3}{\tau_s^{-2}} \left(4\tau_s^{-8} + \varkappa_s^2 \tau_s^{-6} - \frac{1}{2}\tau_s^2 \varkappa_s^2 \tau_s^{-4} - \frac{3}{8}\tau_s^2 \varkappa_s^4 \tau_s^{-2} + \frac{1}{4}\tau_s^4 \varkappa_s^4\right) , \tag{5.15}$$

$$B_1 = \frac{\tau_s\left(\bar{\tau}_s^{2} + \frac{\varkappa_s^{2}}{4}\right)}{\bar{\tau}_s^{2}}\left[6\bar{\tau}_s^{10} - \frac{\varkappa_s^{2}}{2}\,\bar{\tau}_s^{8} + \varkappa_s^{2}\left(\tau_s^{2} - \frac{9}{8}\varkappa_s^{2}\right)\bar{\tau}_s^{6}\right.$$

$$\left. + \frac{5}{8}\varkappa_s^{4}\left(\tau_s^{2} - \frac{\varkappa_s^{2}}{4}\right)\bar{\tau}_s^{4} + \frac{5}{32}\tau_s^{2}\varkappa_s^{6}\bar{\tau}_s^{2} - \frac{1}{16}\,\tau_s^{4}\varkappa_s^{6}\right], \tag{5.16}$$

$$B_2 = -\left(\bar{\tau}_s^{2} + \frac{\varkappa_s^{2}}{4}\right)\left[2\bar{\tau}_s^{8} + \frac{3}{2}\varkappa_s^{2}\bar{\tau}_s^{6} + \frac{\varkappa_s^{2}}{4}\,(7\tau_s^{2} + \varkappa_s^{2})\bar{\tau}_s^{4}\right.$$

$$\left. - \frac{11}{16}\tau_s^{2}\varkappa_s^{4}\bar{\tau}_s^{2} + \frac{3}{8}\tau_s^{4}\varkappa_s^{4}\right], \tag{5.17}$$

$$B_3 = -\,\bar{\tau}_s^{4}\tau_s\left[5\bar{\tau}_s^{4} + \frac{\varkappa_s^{2}}{2}\,\bar{\tau}_s^{2} + \frac{\varkappa_s^{2}}{2}\left(\tau_s^{2} - \frac{7}{8}\varkappa_s^{2}\right)\right], \tag{5.18}$$

$$B_4 = \frac{\bar{\tau}_s^{4}\varkappa_s^{2}}{4}\,(2\bar{\tau}_s^{2} + 3\tau_s^{2}), \tag{5.19}$$

$$B_5 = \bar{\tau}_s^{6}\tau_s\,. \tag{5.20}$$

With non-vanishing of θ_{bs}, $\varkappa_s$, and D, the second polynomial is obtained as follows:

$$\sum_{i=0}^{5} B_i\, x^i = 0\,. \tag{5.21}$$

From (4.23), which is a relation obtained by applying the commutation formula (2.31) to θ_{bs}, on substituting for $\frac{\delta^2\theta_{bs}}{\delta s\delta n}$ the expression obtained by taking the $\underset{\sim}{s}$-gradient of (4.28), and substituting for $\frac{\delta\theta_{bs}}{\delta s}$, $\frac{\delta\tau_s}{\delta s}$, $\frac{\delta\varkappa_s}{\delta s}$, and $\frac{\delta}{\delta s}$ div $\underset{\sim}{b}$, the expressions (4.27), (4.30), (4.33), and (4.36), respectively, and then substituting for $\frac{\delta\theta_{bs}}{\delta n}$ and $\frac{\delta\theta_{bs}}{\delta b}$, the expressions (4.28) and (4.29), respectively, one obtains

$$\frac{\tau_s}{\varkappa_s D^3} \sum_{i=0}^{5} B_i x^i = 0 \ , \tag{5.22}$$

which, with non-vanishing of $\varkappa_s$, τ_s, and D, gives again (5.21). This not only verifies (5.21) but also gives a check on (4.24).

The last polynomial one can obtain is derived and verified as follows.

Applying the commutation formula (2.29) to τ_s, and substituting for $\varkappa_s + \operatorname{div} \underset{\sim}{n}$ the expression (4.1),

$$\frac{\delta^2 \tau_s}{\delta b \delta n} - \frac{\delta^2 \tau_s}{\delta n \delta b} = - \operatorname{div} \underset{\sim}{b} \frac{\delta \tau_s}{\delta n} - \left[\frac{\tau_s}{\theta_{bs}} \operatorname{div} \underset{\sim}{b} - \frac{1}{\varkappa_s}\left(\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\right] \frac{\delta \tau_s}{\delta b} \tag{5.23}$$

Taking the $\underset{\sim}{b}$-gradient of (4.31),

$$\begin{aligned} \frac{\delta^2 \tau_s}{\delta b \delta n} = \frac{\delta^2 \theta_{bs}}{\delta b^2} &+ \left[\frac{2}{\theta_{bs}^2} (\tau_s^2 - 2\tau_s^2) \operatorname{div} \underset{\sim}{b} - \frac{4\tau_s \theta_{bs}}{\varkappa_s}\right] \frac{\delta \theta_{bs}}{\delta b} \\ &+ \left[\frac{4\tau_s}{\theta_{bs}} \operatorname{div} \underset{\sim}{b} - \frac{2}{\varkappa_s}\left(\tau_s^2 + 2\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\right] \frac{\delta \tau_s}{\delta b} \\ &+ \frac{2\tau_s^2}{\theta_{bs}} \frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} + \frac{2\tau_s}{\varkappa_s^2}\left(\tau_s^2 + \frac{\varkappa_s^2}{4}\right) \frac{\delta \varkappa_s}{\delta b} \ . \end{aligned} \tag{5.24}$$

Taking the $\underset{\sim}{n}$-gradient of (4.32),

$$\frac{\delta^2 \tau_s}{\delta n \delta b} = - \frac{\delta^2 \theta_{bs}}{\delta n^2} - \frac{2}{\varkappa_s}\left(\tau_s^2 + 2\theta_{bs}^2 - \frac{\varkappa_s^2}{4}\right)\frac{\delta \theta_{bs}}{\delta n}$$

$$- \frac{4\tau_s \theta_{bs}}{\varkappa_s}\frac{\delta \tau_s}{\delta n} + \frac{2\theta_{bs}}{\varkappa_s^2}\left(\tau_s^2 + \frac{\varkappa_s^2}{4}\right)\frac{\delta \varkappa_s}{\delta n} \quad . \tag{5.25}$$

From (5.23), (5.24), and (5.25), on substituting for $\frac{\delta \tau_s}{\delta n}$, $\frac{\delta \tau_s}{\delta b}$, $\frac{\delta \varkappa_s}{\delta n}$, $\frac{\delta \varkappa_s}{\delta b}$, and $\frac{\delta}{\delta b}$ div $\underset{\sim}{b}$, the expressions (4.31), (4.32), (4.34), (4.35), and (4.38), respectively, one has

$$\frac{\delta^2 \theta_{bs}}{\delta n^2} + \frac{\delta^2 \theta_{bs}}{\delta b^2} = \left[\frac{5\tau_s}{\theta_{bs}} \operatorname{div} \underset{\sim}{b} - \frac{1}{\varkappa_s}\left(5\tau_s^2 + 8\tau_s^2 - \frac{5}{4}\varkappa_s^2\right)\right]\frac{\delta \theta_{bs}}{\delta n}$$

$$- \left(5 \operatorname{div} \underset{\sim}{b} - \frac{\delta \tau_s \theta_{bs}}{\varkappa_s}\right)\frac{\delta \theta_{bs}}{\delta b} - \frac{4\tau_s^2}{\theta_{bs}}(\operatorname{div} \underset{\sim}{b})^2$$

$$- \frac{8\tau_s}{\varkappa_s}\left(2\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\operatorname{div} \underset{\sim}{b}$$

$$+ \frac{4\theta_{bs}}{\varkappa_s^2}\left[\tau_s^4 + 4\tau_s^2\tau_s^2 - \varkappa_s^2\left(\tau_s^2 - \frac{\varkappa_s}{16}\right)\right] . \tag{5.26}$$

From (5.26), on substituting for $\frac{\delta^2 \theta_{bs}}{\delta n^2}$ and $\frac{\delta^2 \theta_{bs}}{\delta b^2}$, the expressions obtained by taking the $\underset{\sim}{n}$ and $\underset{\sim}{b}$ gradients of (4.27) and (4.28), respectively, and substituting for $\frac{\delta \tau_s}{\delta n}$, $\frac{\delta \tau_s}{\delta b}$, $\frac{\delta \varkappa_s}{\delta n}$, $\frac{\delta \varkappa_s}{\delta b}$, $\frac{\delta}{\delta n}$ div $\underset{\sim}{b}$, and $\frac{\delta}{\delta b}$ div $\underset{\sim}{b}$, the expressions (4.31), (4.32), (4.34), (4.35), (4.37), and (4.38), respectively, and then substituting for $\frac{\delta \theta_{bs}}{\delta n}$ and $\frac{\delta \theta_{bs}}{\delta b}$, the expressions (4.28) and (4.29), respectively, one obtains

$$\frac{\theta_{bs}}{\varkappa_s^2 D^3} \sum_{i=0}^{5} C_i \, x^i = 0 \tag{5.27}$$

where

$$C_0 = - \frac{4\tau_s\left(\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}\right)^2}{\tau_{\bar{s}}^2} \left[\tau_{\bar{s}}^{10} + \frac{\varkappa_s^2}{4}\,\tau_{\bar{s}}^8 - \frac{\varkappa_s^2}{2}\left(\tau_s^2 + \frac{\varkappa_s^2}{8}\right)\tau_{\bar{s}}^6 \right.$$

$$\left. - \frac{3}{16}\,\tau_s^2\varkappa_s^4\tau_{\bar{s}}^4 + \frac{\tau_s^2\varkappa_s^8}{256}\right] \tag{5.28}$$

$$C_1 = - \frac{\tau_{\bar{s}}^2 + \frac{\varkappa_s^2}{4}}{\tau_{\bar{s}}^2} \left[4\tau_{\bar{s}}^{12} + (8\tau_s^2 - \varkappa_s^2)\tau_{\bar{s}}^{10} + \varkappa_s^2\left(2\tau_s^2 - \frac{5}{4}\varkappa_s^2\right)\tau_{\bar{s}}^8 \right.$$

$$+ \varkappa_s^2\left(4\tau_s^4 + \frac{1}{2}\tau_s^2\varkappa_s^2 - \frac{3}{16}\varkappa_s^4\right)\tau_{\bar{s}}^6 - \frac{3}{4}\tau_s^2\varkappa_s^4\left(\tau_s^2 - \frac{\varkappa_s^2}{2}\right)\tau_{\bar{s}}^4$$

$$\left. - \frac{\tau_s^2\varkappa_s^6}{4}\left(\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\tau_{\bar{s}}^2 - \frac{\tau_s^4\varkappa_s^8}{64}\right] \tag{5.29}$$

$$C_2 = - \tau_s\left[6\tau_{\bar{s}}^{10} + \frac{3}{2}\varkappa_s^2\tau_{\bar{s}}^8 - \varkappa_s^2\left(\tau_s^2 - \frac{13}{8}\varkappa_s^2\right)\tau_{\bar{s}}^6 - \frac{\varkappa_s^4}{4}\left(11\tau_s^2 - \frac{21}{8}\varkappa_s^2\right)\tau_{\bar{s}}^4 \right.$$

$$\left. - \frac{\varkappa_s^6}{8}\left(5\tau_s^2 - \frac{\varkappa_s^2}{2}\right)\tau_{\bar{s}}^2 - \frac{\tau_s^2\varkappa_s^6}{8}\left(\tau_s^2 + \frac{\varkappa_s^2}{4}\right)\right] \tag{5.30}$$

$$C_3 = 2\tau_{\bar{s}}^{10} + (4\tau_s^2 - 2\varkappa_s^2)\tau_{\bar{s}}^8 + \frac{\varkappa_s^2}{2}\left(11\tau_s^2 - \frac{7}{4}\varkappa_s^2\right)\tau_{\bar{s}}^6 + \frac{\varkappa_s^4}{8}\left(11\tau_s^2 - \frac{\varkappa_s^2}{2}\right)\tau_{\bar{s}}^4$$

$$+ \frac{\tau_s^2\varkappa_s^4}{4}\left(3\tau_s^2 + \frac{\varkappa_s^2}{2}\right)\tau_{\bar{s}}^2 + \frac{\tau_s^4\varkappa_s^6}{16} \tag{5.31}$$

$$C_4 = \tau_{\bar{s}}^{2}\tau_s\left[4\tau_{\bar{s}}^{6} + \frac{3}{2}\varkappa_s^{2}\tau_{\bar{s}}^{4} + \varkappa_s^{2}\left(\tau_s^{2} + \frac{3}{8}\varkappa_s^{2}\right)\tau_{\bar{s}}^{2} + \frac{3}{8}\tau_s^{2}\varkappa_s^{4}\right] \tag{5.32}$$

$$C_5 = \frac{1}{2}\,\tau_{\bar{s}}^{4}\varkappa_s^{2}(\tau_{\bar{s}}^{2} + \tau_s^{2}) \tag{5.33}$$

With θ_{bs}, $\varkappa_s$, and D being different from zero, (5.27) gives the third polynomial

$$\sum_{i=0}^{5} C_i\, x^i = 0 \tag{5.34}$$

Since θ_{bs} and τ_s are each non-vanishing, one may write (5.12) and (5.13) in the following forms:

$$\begin{aligned}
\frac{\delta^2\theta_{bs}}{\delta b^2} &= -\frac{\theta_{bs}}{\tau_s}\frac{\delta^2\theta_{bs}}{\delta b\delta n} + \left(\frac{1}{\theta_{bs}}\frac{\delta\theta_{bs}}{\delta b} - \frac{1}{\tau_s}\frac{\delta\tau_s}{\delta b}\right)\frac{\delta\theta_{bs}}{\delta b} \\
&\quad - \left[\frac{2\tau_{\bar{s}}^{2}\varkappa_s}{\tau_s\theta_{bs}^{2}} - \frac{1}{\theta_{bs}}\left(2\tau_{\bar{s}}^{2} - \frac{\varkappa_s^{2}}{4}\right)\right]\frac{\delta}{\delta b}\operatorname{div}\underset{\sim}{b} - \left[\frac{\varkappa_s}{\tau_s\theta_{bs}^{3}}(\tau_{\bar{s}}^{2} + 2\tau_s^{2})(\operatorname{div}\underset{\sim}{b})^2 \right.\\
&\quad \left. - \frac{1}{\theta_{bs}^{2}}\left(2\tau_s^{2} - \frac{\varkappa_s^{2}}{4}\right)\operatorname{div}\underset{\sim}{b} + \frac{2\theta_{bs}}{\tau_s\varkappa_s}\left(\tau_{\bar{s}}^{2} + 2\theta_{bs}^{2}\frac{\varkappa_s^{2}}{4}\right)\right]\frac{\delta\theta_{\varkappa}}{\delta b} \\
&\quad + \left[\frac{2\varkappa_s}{\theta_{bs}^{2}}(\operatorname{div}\underset{\sim}{b})^2 - \frac{1}{\tau_s\theta_{bs}}\left(2\tau_{\bar{s}}^{2} + 4\tau_s^{2} - \frac{\varkappa_s^{2}}{4}\right)\operatorname{div}\underset{\sim}{b} - \frac{4\theta_{bs}^{2}}{\varkappa_s}\right]\frac{\delta\tau_s}{\delta b} \\
&\quad + \left[\frac{\tau_s^{2}}{\tau_s\theta_{bs}^{2}}(\operatorname{div}\underset{\sim}{b})^2 + \frac{\varkappa_s}{2\theta_{bs}}\operatorname{div}\underset{\sim}{b} - \frac{2\theta_{bs}^{2}}{\tau_s\varkappa_s^{2}}\left(\tau_{\bar{s}}^{2} + \frac{\varkappa_s^{2}}{4}\right)\right]\frac{\delta\varkappa_s}{\delta b}
\end{aligned} \tag{5.35}$$

and

$$\frac{\delta^2\theta_{bs}}{\delta n^2} = -\frac{\tau_s}{\theta_{bs}}\frac{\delta^2\theta_{bs}}{\delta n\delta b} + \left(\frac{1}{\tau_s}\frac{\delta\tau_s}{\delta n} - \frac{1}{\theta_{bs}}\frac{\delta\theta_{bs}}{\delta n}\right)\frac{\delta\theta_{bs}}{\delta n}$$

$$- \left[\frac{2\tilde{\tau}_s^{\,2}\varkappa}{\theta_{bs}^{\,3}}\,\mathrm{div}\,\underset{\sim}{b} - \frac{\tau_s}{\theta_{bs}^{\,2}}\left(2\tilde{\tau}_s^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right)\right]\frac{\delta}{\delta n}\,\mathrm{div}\,\underset{\sim}{b}$$

$$+ \left[\frac{2\tau_s^{\,2}\varkappa_s}{\theta_{bs}^{\,4}}\,(\mathrm{div}\,\underset{\sim}{b})^2 + \frac{\tau_s}{\theta_{bs}^{\,3}}\left(2\tilde{\tau}_s^{\,2} - 4\theta_{bs}^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right)\mathrm{div}\,\underset{\sim}{b}\right.$$

$$\left. - \frac{4}{\varkappa_s}\left(\tilde{\tau}_s^{\,2} + \theta_{bs}^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right)\right]\frac{\delta\theta_{bs}}{\delta n} - \left[\frac{\tilde{\tau}_s^{\,2}\varkappa_s}{\tau_s\theta_{bs}^{\,3}}\,(\mathrm{div}\,\underset{\sim}{b})^2 + \frac{4\tau_s^{\,2}}{\theta_{bs}^{\,2}}\,\mathrm{div}\,\underset{\sim}{b}\right.$$

$$\left. - \frac{2\theta_{bs}}{\tau_s\varkappa_s}\left(\tilde{\tau}_s^{\,2} - 2\tau_s^{\,2} - \frac{\varkappa_s^{\,2}}{4}\right)\right]\frac{\delta\tau_s}{\delta n}$$

$$+ \left[\frac{\tilde{\tau}_s^{\,2}}{\theta_{bs}^{\,3}}\,(\mathrm{div}\,\underset{\sim}{b})^2 + \frac{\tau_s\varkappa_s}{2\theta_{bs}^{\,2}}\,\mathrm{div}\,\underset{\sim}{b} + \frac{2\theta_{bs}}{\varkappa_s^{\,2}}\left(\tilde{\tau}_s^{\,2} + \frac{\varkappa_s^{\,2}}{4}\right)\right]\frac{\delta\varkappa_s}{\delta n} \qquad (5.36)$$

Adding (5.35) to (5.36), on substituting for $\frac{\delta^2\theta_{bs}}{\delta b\delta n}$ and $\frac{\delta^2\theta_{bs}}{\delta n\delta b}$, the expressions obtained by taking the b and n gradients of (4.28) and (4.29), respectively, and substituting for $\frac{\delta\tau_s}{\delta n}$, $\frac{\delta\tilde{\tau}_s}{\delta b}$, $\frac{\delta\varkappa_s}{\delta n}$, $\frac{\delta\varkappa_s}{\delta b}$, $\frac{\delta}{\delta n}\,\mathrm{div}\,\underset{\sim}{b}$, $\frac{\delta}{\delta b}\,\mathrm{div}\,\underset{\sim}{b}$, and $\frac{\delta^2\theta_{bs}}{\delta n^2} + \frac{\delta^2\theta_{bs}}{\delta b^2}$, the expressions (4.31) (4.32), (4.34), (4.35), (4.37), (4.38), and (5.26), respectively, and then substituting for $\frac{\delta\theta_{bs}}{\delta n}$ and $\frac{\delta\theta_{bs}}{\delta b}$ the expressions (4.28) and (4.29), respectively, one again obtains (5.27). The polynomial (5.34) is thus verified.

An analysis given in the Appendix will show that polynomials (5.11), (5.21), and (5.34) are the only such relations obtainable for the vector-field in question, from the basic compatibility conditions (2.39) to (2.47).

CHAPTER VI

A MAIN LEMMA

With the relations (4.28) and (4.29) for $\frac{\delta\theta_{bs}}{\delta n}$ and $\frac{\delta\theta_{bs}}{\delta b}$ at hand, we prove the following main lemma.

Lemma 6.1.

The existence of two functional relationships,

$$F(\theta_{bs}, \varkappa_s) = 0 \tag{6.1}$$

$$G(\tau_s, \varkappa_s) = 0 \tag{6.2}$$

is a sufficient condition for Hamel's Theorem to hold.

From the relationship (6.1), one has*

$$\text{grad } \varkappa_s \times \text{grad } \theta_{bs} = 0 \ . \tag{6.3}$$

One consequence of (6.3) is the relation

$$\frac{\delta\varkappa_s}{\delta s}\frac{\delta\theta_{bs}}{\delta b} - \frac{\delta\varkappa_s}{\delta b}\frac{\delta\theta_{bs}}{\delta s} = 0 \ . \tag{6.4}$$

Substituting from (4.33), (4.29), (4.35), and (4.27) in (6.4), and reducing, one has, since θ_{bs} and $\varkappa_s$ are non-vanishing,

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*See reference (1947 [1]).

$$\tau_{\bar{s}}^2 x^2 + \frac{\tau_s \kappa_s^2}{2}\, x - 2\left(\tau_{\bar{s}}^2 + \frac{\kappa_s^2}{4}\right)^2 = 0 , \qquad (6.5)$$

where x is defined by (5.3).

From the relationship (6.2), one has

$$\operatorname{grad} \kappa_s \times \operatorname{grad} \tau_s = 0 , \qquad (6.6)$$

one consequence of which is the condition

$$\frac{\delta \kappa_s}{\delta s}\frac{\delta \tau_s}{\delta b} - \frac{\delta \kappa_s}{\delta b}\frac{\delta \tau_s}{\delta s} = 0 . \qquad (6.7)$$

From (6.7), on substituting for $\frac{\delta \kappa_s}{\delta s}$, $\frac{\delta \tau_s}{\delta b}$, $\frac{\delta \kappa_s}{\delta b}$, and $\frac{\delta \tau_s}{\delta s}$, the expressions (4.33), (4.32), (4.35), and (4.30), respectively, and then substituting for $\frac{\delta \theta_{bs}}{\delta n}$ the expression (4.28) and reducing, one obtains

$$\tau_s \tau_{\bar{s}}^2 x^2 - \left[4\tau_{\bar{s}}^4 - \frac{\kappa_s^2}{2}\left(2\tau_{\bar{s}}^2 + \tau_s^2\right)\right]x$$

$$- 2\tau_s\left(3\tau_{\bar{s}}^4 + \frac{\kappa_s^2}{2}\tau_{\bar{s}}^2 - \frac{\kappa_s^4}{16}\right) = 0 . \qquad (6.8)$$

Since τ_s is non-vanishing, one may obtain, on multiplying (6.5) by τ_s, and subtracting from (6.8),

$$D\left(\tau_{\bar{s}}^2 - \frac{\kappa_s^2}{4}\right) = 0 , \qquad (6.9)$$

where D is defined by (3.16), and is non-vanishing by virtue of Lemma 3.5. Hence, from (6.9), one has

$$\tau_{\bar{s}}^{2} - \frac{\varkappa_{s}^{2}}{4} = 0 \quad . \tag{6.10}$$

Taking the $\underset{\sim}{s}$-gradient of (6.10), and substituting for $\frac{\delta \tau_{\bar{s}}^{2}}{\delta s}$ and $\frac{\delta \varkappa_{s}}{\delta s}$, the expressions (2.55) and (4.33), one obtains

$$\varkappa_{s} \, \theta_{bs} = 0 \quad . \tag{6.11}$$

Thus either $\varkappa_{s}$ or θ_{bs} vanishes. Accordingly (6.1) and (6.2) ensure that Hamel's theorem holds.

CHAPTER VII

CONDITIONAL PROOF OF HAMEL'S THEOREM

The relations (5.11), (5.21), and (5.34) are three polynomial equations in the four variables θ_{bs}, τ_s, $\varkappa_s$, and div $\underset{\sim}{b}$.

If one can assert that these three equations are equivalent to relations

$$F(\tau_s, \varkappa_s) = 0 \qquad (7.1)$$

$$G(\theta_{bs}, \varkappa_s) = 0 \qquad (7.2)$$

then Hamel's theorem follows as a result of the Lemma established in the preceding section.

One may apply Sylvester's method (1928 [1]) to obtain the eliminants of the polynomials in pairs, in the form of determinants:

$$D_1(\theta_{bs},\tau_s,\varkappa_s) = \begin{vmatrix} B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 & 0 & 0 & 0 \\ 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 & 0 & 0 \\ 0 & 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 & 0 \\ 0 & 0 & 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 \\ 0 & 0 & 0 & 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 \\ C_5 & C_4 & C_3 & C_2 & C_1 & C_0 & 0 & 0 & 0 & 0 \\ 0 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 & 0 & 0 & 0 \\ 0 & 0 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 & 0 & 0 \\ 0 & 0 & 0 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 \end{vmatrix} \qquad (7.3)$$

$$D_2(\theta_{bs},\tau_s,\varkappa_s) = \begin{vmatrix} A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & 0 & 0 & 0 & 0 \\ 0 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & 0 & 0 & 0 \\ 0 & 0 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & 0 & 0 \\ 0 & 0 & 0 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & 0 \\ 0 & 0 & 0 & 0 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 \\ B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_5 & B_4 & B_3 & B_2 & B_1 & B_0 \end{vmatrix} \qquad (7.4)$$

If these determinants each vanish identically then there is a single relation

$$f(\theta_{bs},\ \tau_s,\ \varkappa_s,\ \mathrm{div}\ \underset{\sim}{b}) = 0 \qquad (7.5)$$

causing (5.11), (5.21), and (5.34) to be satisfied simultaneously. In this eventuality ones only recourse would be to try to establish a contradiction by analyzing (7.5) with Hamel's condition (2.53) in the manner of the proof of Lemma 3.5.

If the determinants do not vanish identically so that the relations do not possess a common factor, then one has the two equations

$$D_1(\theta_{bs},\ \tau_s,\ \varkappa_s) = 0 \qquad (7.6)$$

$$D_2(\theta_{bs},\ \tau_s,\ \varkappa_s) = 0 \quad . \qquad (7.7)$$

If we eliminate θ_{bs} from these, one would obtain a high order determinant $F(\tau_s, \varkappa_s)$. If this vanishes identically then there exists a relation

$$g(\theta_{bs}, \tau_s, \varkappa_s) = 0 \tag{7.8}$$

This again with Hamel's condition might prove to be an exceptional field.

Otherwise, if the determinant does not vanish identically, one has the equation

$$F(\tau_s, \varkappa_s) = 0 \quad . \tag{7.9}$$

One may use the same method to eliminate τ_s from (6.6) and (6.7) to obtain

$$G(\theta_{bs}, \varkappa_s) = 0 \quad . \tag{7.10}$$

If this is possible, then Hamel's theorem follows from the main lemma.

The complexity and essential difference in the three polynomials makes it hard to believe this deduction is not possible.

APPENDIX

If one applies the commutation formulae (2.29), (2.30), and (2.31) to each of the parameters θ_{bs}, τ_s, $\varkappa_s$, and div $\underset{\sim}{b}$. One will obtain what appears to be twelve relations. In this Appendix we demonstrate that the conditions (4.22), (4.23), (5.1), and (5.26), which reduce to the three polynomials, are the only four new relations obtainable from these twelve relations. Hence, the three polynomials are the only such conditions obtainable in this way.

Since (4.22), (4.23), and (5.1) are obtained by applying the commutation formulae (2.30), (2.31), and (2.29) to θ_{bs}, respectively, and (5.26) is obtained by applying the commutation formula (2.29) to τ_s, one proves this by investigating the remaining eight relations one by one as follows:

Applying (2.30) to τ_s, one has

$$\frac{\delta^2 \tau_s}{\delta s \delta b} - \frac{\delta^2 \tau_s}{\delta b \delta s} = - \theta_{bs} \frac{\delta \tau_s}{\delta b} . \tag{A.1}$$

From (A.1), on substituting for $\frac{\delta^2 \tau_s}{\delta s \delta b}$ and $\frac{\delta^2 \tau_s}{\delta b \delta s}$ the expressions obtained by taking the $\underset{\sim}{s}$ and $\underset{\sim}{b}$ gradients of (4.32) and (4.30), respectively, and substituting for, $\frac{\delta \tau_s^2}{\delta s}$, $\frac{\delta \theta_{bs}}{\delta s}$, $\frac{\delta \tau_s}{\delta b}$, $\frac{\delta \varkappa_s}{\delta s}$, $\frac{\delta \varkappa_s}{\delta b}$, and $\frac{\delta}{\delta b}$ div $\underset{\sim}{b}$, the expressions (2.55), (4.27), (4.32), (4.33), (4.35), and (4.38), respectively, one obtains the relation (4.23). Similarly, the application of the commutation formula (2.31) to τ_s, will yield the relation (4.22), due to the condition (4.31).

By the same process one may show that when the formulae (2.29) and (2.30) are applied to $\varkappa_s$, only identities result.

If one applies (2.31) to $\varkappa_s$, and follows the same process, one will again obtain the relation (4.17). The relation (4.17) is thus verified.

By applying (2.29), (2.30), and (2.31) to div $\underset{\sim}{b}$ one obtains only algebraic combinations of the relations (4.20), (4.21), (5.1), and (5.26). We show this as follows:

By (2.29), one has

$$\frac{\delta^2}{\delta b \delta n} \operatorname{div} \underset{\sim}{b} - \frac{\delta^2}{\delta n \delta b} \operatorname{div} \underset{\sim}{b}$$

$$= - \operatorname{div} \underset{\sim}{b} \frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} + (\varkappa_s + \operatorname{div} \underset{\sim}{n}) \frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} \quad . \qquad \text{(A.2)}$$

Taking the $\underset{\sim}{b}$ and $\underset{\sim}{n}$ gradients of (4.37) and (4.38) respectively, one obtains

$$\frac{\delta^2}{\delta b \delta n} \operatorname{div} \underset{\sim}{b} = \left(\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} + \frac{4 \tau_s \theta_{bs}^{\;2}}{\tau_{\bar{s}}^{\;2} \varkappa_s} \right) \frac{\delta^2 \theta_{bs}}{\delta b \delta n} - \frac{2 \theta_{bs}}{\tau_{\bar{s}}^{\;2} \varkappa_s} (\tau_{\bar{s}}^{\;2} - 2\tau_s^{\;2}) \frac{\delta^2 \theta_{bs}}{\delta b^2}$$

$$+ \left[\frac{1}{\theta_{bs}} \frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} - \left(\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} - \frac{8 \tau_s^{\;2} \theta_{bs}}{\tau_{\bar{s}}^{\;4} \varkappa_s} \right) \frac{\delta \theta_{bs}}{\delta b} \right.$$

$$\left. + \frac{4 \theta_{bs}^{\;2}}{\tau_{\bar{s}}^{\;4} \varkappa_s} (\tau_{\bar{s}}^{\;2} - 2\tau_s^{\;2}) \frac{\delta \tau_s}{\delta b} - \frac{4 \tau_s \theta_{bs}^{\;2}}{\tau_{\bar{s}}^{\;2} \varkappa_s^{\;2}} \frac{\delta \varkappa_s}{\delta b} \right] \frac{\delta \theta_{bs}}{\delta n}$$

$$+ \left[\frac{2 \theta_{bs}}{\tau_{\bar{s}}^{\;2} \varkappa_s^{\;2}} (\tau_{\bar{s}}^{\;2} - 2\tau_s^{\;2}) \frac{\delta \varkappa_s}{\delta b} \right.$$

$$- \frac{2}{\tau_{\bar{s}}^{4}\kappa_s}\left(\tau_{\bar{s}}^{4} - 2\tau_s^{2}\tau_{\bar{s}}^{2} + 4\tau_s^{2}\theta_{bs}^{2}\right)\frac{8\theta_{bs}}{\delta b} + \frac{\delta\tau_s\theta_{bs}^{3}}{\tau_{\bar{s}}^{4}\kappa_s}\frac{\delta\tau_s}{\delta b}\Bigg]\frac{\delta\theta_{bs}}{\delta b}$$

$$+ \cdots\cdots \tag{A.3}$$

$$\frac{\delta^2}{\delta n\delta b}\operatorname{div}\underset{\sim}{b} = -\frac{2\theta_{bs}}{\tau_{\bar{s}}^{2}\kappa_s}\left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right)\frac{\delta^2\theta_{bs}}{\delta n^2} + \left(\frac{\operatorname{div}\underset{\sim}{b}}{\theta_{bs}} - \frac{4\tau_s\theta_{bs}^{2}}{\tau_{\bar{s}}^{2}\kappa_s}\right)\frac{\delta^2\theta_{bs}}{\delta n\delta b}$$

$$+ \Bigg[\frac{2\theta_{bs}}{\tau_{\bar{s}}^{2}\kappa_s^{2}}\left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right)\frac{\delta\kappa_s}{\delta n}$$

$$- \frac{2}{\tau_{\bar{s}}^{4}\kappa_s}\left(\tau_{\bar{s}}^{4} - 2\tau_s^{2}\tau_{\bar{s}}^{2} + 4\tau_s^{2}\theta_{bs}^{2}\right)\frac{\delta\theta_{bs}}{\delta n} + \frac{8\tau_s\theta_{bs}^{3}}{\tau_{\bar{s}}^{4}\kappa_s}\frac{\delta\tau_s}{\delta n}\Bigg]\frac{\delta\theta_{bs}}{\delta n}$$

$$+ \Bigg[\frac{1}{\theta_{bs}}\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} - \left(\frac{\operatorname{div}\underset{\sim}{b}}{\theta_{bs}^{2}} + \frac{8\tau_s^{3}\theta_{bs}}{\tau_s^{4}\kappa_s}\right)\frac{\delta\theta_{bs}}{\delta n}$$

$$- \frac{4\theta_{bs}^{2}}{\tau_{\bar{s}}^{4}\kappa_s}\left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right)\frac{\delta\tau_s}{\delta n} + \frac{4\tau_s\theta_{bs}^{2}}{\tau_{\bar{s}}^{2}\kappa_s^{2}}\frac{\delta\kappa_s}{\delta n}\Bigg]\frac{\delta\theta_{bs}}{\delta b}$$

$$+ \cdots\cdot \tag{A.4}$$

Since the full expressions of (A.3) and (A.4) are very long, for convenience, only the terms containing second gradients or quadratic of first gradient are shown.

From (A.2), (A.3) and (A.4), on substituting for $\frac{\delta\tau_s}{\delta n}$, $\frac{\delta\tau_s}{\delta b}$, $\frac{\delta\kappa_s}{\delta n}$, $\frac{\delta\kappa_s}{\delta b}$, $\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b}$, and $\frac{\delta}{\delta b}\operatorname{div}\underset{\sim}{b}$, the expressions (4.31), (4.32), (4.34), (4.35), (4.37), and (4.38), respectively, and reducing, one obtains

$$\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} \left(\frac{\delta^2\theta_{bs}}{\delta b \delta n} - \frac{\delta^2\theta_{bs}}{\delta n \delta b}\right) + \frac{2\theta_{bs}}{\tau_{\bar{s}}^{2} \varkappa_s} \left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right) \left(\frac{\delta^2\theta_{bs}}{\delta n^2} - \frac{\delta^2\theta_{bs}}{\delta b^2}\right)$$

$$+ \frac{4\tau_s \theta_{bs}^{2}}{\tau_{\bar{s}}^{2} \varkappa_s} \left(\frac{\delta^2\theta_{bs}}{\delta b \delta n} + \frac{\delta^2\theta_{bs}}{\delta n \delta b}\right) - \frac{4\theta_{bs}^{2}}{\tau_{\bar{s}}^{4} \varkappa_s} \left(\tau_{\bar{s}}^{2} - 4\tau_s^{2}\right)\left[\left(\frac{\delta\theta_{bs}}{\delta n}\right)^2 - \left(\frac{\delta\theta_{bs}}{\delta b}\right)^2\right]$$

$$- \frac{16\tau_s \theta_{bs}}{\tau_{\bar{s}}^{4} \varkappa_s} \left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right) \frac{\delta\theta_{bs}}{\delta n} \frac{\delta\theta_{bs}}{\delta b} + \cdots = 0 \,. \tag{A.5}$$

On substituting for $\frac{\delta^2\theta_{bs}}{\delta b \delta n} + \frac{\delta^2\theta_{bs}}{\delta n \delta b}$ the expression obtained by adding (5.12) to (5.13), and substituting for $\frac{\delta\tau_s}{\delta n}$, $\frac{\delta\tau_s}{\delta b}$, $\frac{\delta\varkappa_s}{\delta n}$, $\frac{\delta\varkappa_s}{\delta b}$, $\frac{\delta}{\delta n}\operatorname{div} \underset{\sim}{b}$, and $\frac{\delta}{\delta b}\operatorname{div} \underset{\sim}{b}$, the expressions (4.31), (4.32), (4.34), (4.35), (4.37), and (4.38), (A.5) reduces to the following form

$$\frac{\operatorname{div} \underset{\sim}{b}}{\theta_{bs}} \left(\frac{\delta^2\theta_{bs}}{\delta b \delta n} - \frac{\delta^2\theta_{bs}}{\delta n \delta b}\right) - \frac{2\theta_{bs}}{\varkappa_s} \left(\frac{\delta^2\theta_{bs}}{\delta n^2} + \frac{\delta^2\theta_{bs}}{\delta b^2}\right)$$

$$- \left(\theta_{bs} \frac{\delta\theta_{bs}}{\delta n} + \tau_s \frac{\delta\theta_{bs}}{\delta b}\right)\left[\frac{8\theta_{bs}}{\tau_{\bar{s}}^{4} \varkappa_s} \left(\tau_{\bar{s}}^{2} - 2\tau_s^{2}\right) \frac{\delta\theta_{bs}}{\delta n}\right.$$

$$\left. - \frac{4}{\tau_{\bar{s}}^{4} \tau_s \varkappa_s} \left(\tau_{\bar{s}}^{4} - 4\tau_s^{2}\theta_{bs}^{2}\right) \frac{\delta\theta_{bs}}{\delta b}\right] + \cdots = 0 \,. \tag{A.6}$$

A substitution for $\frac{\delta^2\theta_{bs}}{\delta b \delta n} - \frac{\delta^2\theta_{bs}}{\delta n \delta b}$, $\frac{\delta^2\theta_{bs}}{\delta n^2} + \frac{\delta^2\theta_{bs}}{\delta b^2}$, $\theta_{bs} \frac{\delta\theta_{bs}}{\delta n} + \tau_s \frac{\delta\theta_{bs}}{\delta b}$, in the expressions (5.1), (5.26), and (4.18), (A.6) yields an identity.

From (2.30), one has

$$\frac{\delta^2}{\delta s \delta b} \operatorname{div} \underset{\sim}{b} - \frac{\delta^2}{\delta b \delta s} \operatorname{div} \underset{\sim}{b} = - \theta_{bs} \frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} \; . \tag{A.7}$$

For the purpose of this proof, one will find it more convenient to take the $\underset{\sim}{s}$-gradient of (2.47) rather than (4.38), to yield the following expression of $\frac{\delta^2}{\delta s \delta b} \operatorname{div} \underset{\sim}{b}$.

$$\frac{\delta^2}{\delta s \delta b} \operatorname{div} \underset{\sim}{b} = - \frac{\delta^2}{\delta s \delta n} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) + 2 \operatorname{div} \underset{\sim}{b} \frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{b}$$

$$+ 2(\varkappa_s + \operatorname{div} \underset{\sim}{n}) \frac{\delta}{\delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) - \frac{\delta \tau_s^2}{\delta s} \; . \tag{A.8}$$

Taking the $\underset{\sim}{b}$-gradient of (4.36),

$$\frac{\delta^2}{\delta b \delta s} \operatorname{div} \underset{\sim}{b} = - \frac{\delta^2 \theta_{bs}}{\delta b^2} - \frac{\delta \theta_{bs}}{\delta b} \operatorname{div} \underset{\sim}{b} - \theta_{bs} \frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} \; . \tag{A.9}$$

Applying the commutation formula (2.31) to $\varkappa_s + \operatorname{div} \underset{\sim}{n}$,

$$\frac{\delta^2}{\delta s \delta n} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) = \frac{\delta^2}{\delta n \delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) - \varkappa_s \frac{\delta}{\delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n})$$

$$+ \theta_{bs} \frac{\delta}{\delta n} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) + 2\tau_s \frac{\delta}{\delta b} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) \; . \tag{A.10}$$

Taking the $\underset{\sim}{n}$-gradient of (2.48),

$$\frac{\delta^2}{\delta n \delta s} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) = - \frac{\delta^2 \theta_{bs}}{\delta n^2} - (4\varkappa_s + 3 \operatorname{div} \underset{\sim}{n}) \frac{\delta \theta_{bs}}{\delta n} - \theta_{bs} \frac{\delta \varkappa_s}{\delta n}$$

$$- 3\theta_{bs} \frac{\delta}{\delta n} (\varkappa_s + \operatorname{div} \underset{\sim}{n}) - 2 \operatorname{div} \underset{\sim}{b} \frac{\delta \tau_s}{\delta n}$$

$$- 2\tau_s \frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} \; . \tag{A.11}$$

From (A.7), (A.8), (A.9), (A.10), and (A.11), on substituting for $\frac{\delta}{\delta b}\operatorname{div}\underset{\sim}{b}$, $\frac{\delta}{\delta s}\operatorname{div}\underset{\sim}{b}$, $\frac{\delta}{\delta s}(\kappa_s + \operatorname{div}\underset{\sim}{n})$, $\kappa_s + \operatorname{div}\underset{\sim}{n}$, $\frac{\delta\tau_s^{-2}}{\delta s}$, $\frac{\delta}{\delta n}(\kappa_s + \operatorname{div}\underset{\sim}{n})$, $\frac{\delta}{\delta b}(\kappa_s + \operatorname{div}\underset{\sim}{n})$, $\frac{\delta\kappa_s}{\delta n}$, $\frac{\delta\tau_s}{\delta n}$, and $\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b}$, the expressions (4.38), (4.36), (2.48), (4.1), (2.55), (4.15), (4.13), (4.34), (4.31), and (4.37), respectively, one reduces (A.7) to the relation (5.26).

Applying the commutation formula (2.31) to div $\underset{\sim}{b}$,

$$\frac{\delta^2}{\delta n\delta s}\operatorname{div}\underset{\sim}{b} - \frac{\delta^2}{\delta s\delta n}\operatorname{div}\underset{\sim}{b} = \kappa_s\frac{\delta}{\delta s}\operatorname{div}\underset{\sim}{b} - \theta_{bs}\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} - 2\tau_s\frac{\delta}{\delta b}\operatorname{div}\underset{\sim}{b}\,. \tag{A.12}$$

Taking the $\underset{\sim}{n}$-gradient of (3.36),

$$\frac{\delta^2}{\delta n\delta s}\operatorname{div}\underset{\sim}{b} = -\frac{\delta^2\theta_{bs}}{\delta n\delta b} - \frac{\delta\theta_{bs}}{\delta n}\operatorname{div}\underset{\sim}{b} - \theta_{bs}\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b}\,. \tag{A.13}$$

Since κ_s is different from zero, one has from (4.11)

$$\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} = \frac{\delta}{\delta b}(\kappa_s + \operatorname{div}\underset{\sim}{n}) + \frac{4}{\kappa_s}\left(\tau_s\frac{\delta\theta_{bs}}{\delta n} - \theta_{bs}\frac{\delta\theta_{bs}}{\delta b}\right)$$
$$- \frac{4\tau_s^{-2}}{\kappa_s}\operatorname{div}\underset{\sim}{b} + \frac{\delta\tau_s\theta_{bs}}{\kappa_s^2}\left(\tau_s^{-2} - \frac{\kappa_s^2}{4}\right)\,. \tag{A.14}$$

One will also find it more convenient here to take the $\underset{\sim}{s}$-gradient of (A.14) rather than (4.37) to give $\frac{\delta^2}{\delta s\delta n}\operatorname{div}\underset{\sim}{b}$. One obtains the following expression.

$$\frac{\delta^2}{\delta s\delta n}\operatorname{div}\underset{\sim}{b} = \frac{\delta^2}{\delta s\delta b}(\kappa_s + \operatorname{div}\underset{\sim}{n}) + \frac{4}{\kappa_s}\left(\tau_s\frac{\delta^2\theta_{bs}}{\delta s\delta n} - \theta_{bs}\frac{\delta^2\theta_{bs}}{\delta s\delta b}\right) - \frac{4\tau_s^{-2}}{\kappa_s}\frac{\delta}{\delta s}\operatorname{div}\underset{\sim}{b}$$

$$+ \left[\frac{4}{\varkappa_s}\frac{\delta\theta_{bs}}{\delta n} - \frac{8\tau_s}{\varkappa_s}\operatorname{div}\underset{\sim}{b} + \frac{8\theta_{bs}}{\varkappa_s^2}\left(\tau_{\bar{s}}^2 + 2\tau_s^2 - \frac{\varkappa_s^2}{4}\right)\right]\frac{\delta\tau_s}{\delta s}$$

$$- \left[\frac{4}{\varkappa_s^2}\left(\tau_s\frac{\delta\theta_{bs}}{\delta n} - \theta_{bs}\frac{\delta\theta_{bs}}{\delta b}\right) - \frac{4\tau_{\bar{s}}^2}{\varkappa_s^2}\operatorname{div}\underset{\sim}{b} + \frac{16\tau_{\bar{s}}^2\tau_s\theta_{bs}}{\varkappa_s^3}\right]\frac{\delta\varkappa_s}{\delta s}$$

$$- \left[\frac{4}{\varkappa_s}\frac{\delta\theta_{bs}}{\delta b} + \frac{8\theta_{bs}}{\varkappa_s}\operatorname{div}\underset{\sim}{b} - \frac{8\tau_s}{\varkappa_s^2}\left(\tau_{\bar{s}}^2 + 2\theta_{bs}^2 - \frac{\varkappa_s^2}{4}\right)\right]\frac{\delta\theta_{bs}}{\delta s} \quad \text{(A.15)}$$

Applying (2.30) to $\varkappa_s + \operatorname{div}\underset{\sim}{n}$,

$$\frac{\delta^2}{\delta s\delta b}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) = \frac{\delta^2}{\delta b\delta s}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) - \theta_{bs}\frac{\delta}{\delta b}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) \quad \text{(A.16)}$$

Taking the $\underset{\sim}{b}$-gradient of (2.48),

$$\frac{\delta^2}{\delta b\delta s}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) = -\frac{\delta^2\theta_{bs}}{\delta b\delta n} + (4\varkappa_s + 3\operatorname{div}\underset{\sim}{n})\frac{\delta\theta_{bs}}{\delta b} + 2\operatorname{div}\underset{\sim}{b}\frac{\delta\tau_s}{\delta b}$$

$$+ 3\theta_{bs}\frac{\delta}{\delta b}(\varkappa_s + \operatorname{div}\underset{\sim}{n}) + 2\tau_s\frac{\delta}{\delta b}\operatorname{div}\underset{\sim}{b} + \theta_{bs}\frac{\delta\varkappa_s}{\delta s} \quad \text{(A.17)}$$

From (A.12), (A.13), (A.14), (A.15), (A.16), and (A.17), on substituting for $\frac{\delta}{\delta s}\operatorname{div}\underset{\sim}{b}$, $\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b}$, $\frac{\delta}{\delta b}\operatorname{div}\underset{\sim}{b}$, $\frac{\delta^2\theta_{bs}}{\delta s\delta n}$, $\frac{\delta^2\theta_{bs}}{\delta s\delta b}$, $\frac{\delta\tau_s}{\delta s}$, $\frac{\delta\varkappa_s}{\delta s}$, $\frac{\delta\theta_{bs}}{\delta s}$, $\frac{\delta}{\delta b}(\varkappa_s + \operatorname{div}\underset{\sim}{n})$, and $\frac{\delta\tau_s}{\delta b}$, the expressions (4.36), (4.37), (4.38), (4.22), (4.23), (4.30), (4.33), (4.27), (4.13), and (4.32), respectively, one reduces (A.12) to the relation (5.1).

Finally one can conclude here that (4.22), (4.23), (5.1) and (5.26) are the only four available relations.

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